Syllabus

2om

Unit 1] 1-D elements & problems

Unit 2] Truss and Beam elements & pb

Unit 3] Multiconstraint 1-D element & 2-D elements & problems.

Unit 4] Applications and related pb.

Books

FEM by Bailey Grandu

Chandra patla

Scanned by CamScanner

Unit : 1

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1-D Element.

☆ Steps involve in FEM

1) Breaking down a large component or body into small finit element such that grid or mesh of this element covers the entire component. this activity is called as mesh genr or discritization.

2) Eqn governing to individual element are work out & approximate soln to the eqn of an element are find out

3) Soln of individual elements are assembled together considering the connectivity of the elements.

4) Boundary condn are applied to the assembled eqn & final soln is achieve.

5) Soln is post process to workout additional parameter & inforn that is to modify the soln

6) Results are tested for convergence & error estimate for analysis.

☆ Types of Forces in FEM :→

1) Body force.
2) Traction force.
3) Point force.

1) Body force :→ The force which are distributed uniformly over the body are kn. as body force.
Ex:- wt of the body.
It is expressed as force per unit volume. (N/mm^3)

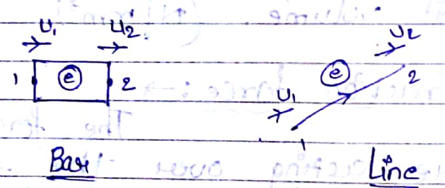
2) Traction force :→ The force which are acting over the surface of the body are called traction force, frictional force & viscous force.
It is expressed as force per unit area. (N/mm^2)

3) Point force / Nodal force :→
The externally applied ~~load~~ force at a point is kn. as point force. They are expressed as a absolute unit. [N]

★ Types of Element

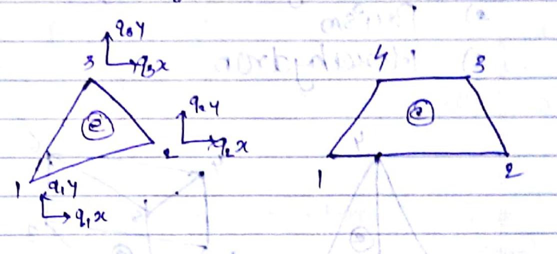
1) 1-D element
2-D
3-D

1) 1-D element →
If the displacement of node expressed only in one dir^n then such element are called as 1-D element.
Ex:- Bar & line element

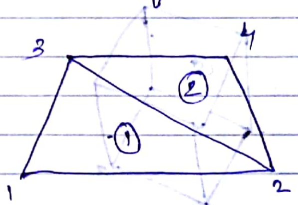


Appli:- Bar element is used in the discretization of 1-D element whereas line element is used for discritization of roof truss

2) 2-D Element →
In 2-D element displacement of each node is expressed in two mutually ⊥^r dir^n called as 2-D element.
Ex:- Triangular & quadrilateral element.



Triangular element is the fundamental element becoz quadrilateral element can be expressed into two combination of triangular element.

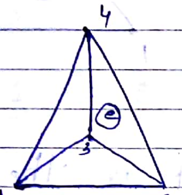


Appl:→ 2-D elements are used in the pb. of plain stresses & strain.

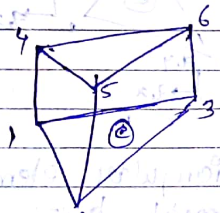
3) **3-D element** →

In 3-D element each nodes having 3 displ. mutually ⊥ to each other. There are 3 types

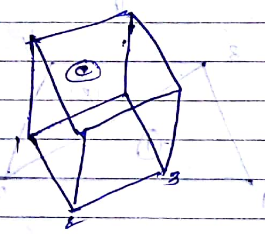
1) Tetrahydron
2) Prism
3) Hexahydron



Tetrahydron (4 nodes)



Prism (6 nodes)



These elements are used in the discitization of solid bodies

☆ **Degree of freedom** :→

It is defined as the no. of dirn in which the displ. of nodes are to be expressed are called DOF of a nodes.

i) In case of 1-D element the DOF associated with node is 1

ii) In case of 2-D element the DOF asso— — node is 2

iii) — — 3D element — 3

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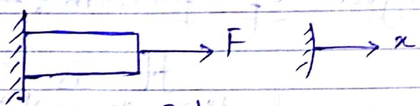
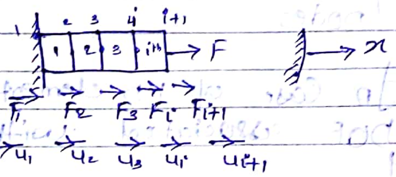
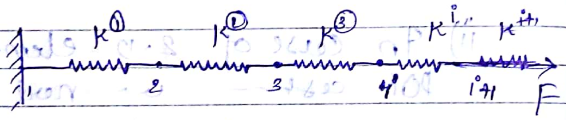
① ★ Derive Element Stiffness matrix for 1-D bar element. (direct approach / conventional)



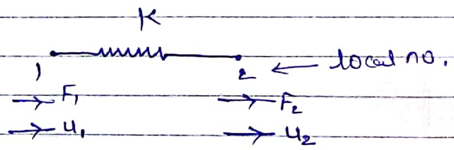
1-D Body

Step ① → Discritization.





Consider having stiffness K with local nodes 1, 2



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When only force F_1 is act in the element than

$$K = \frac{F_1}{(u_1 - u_2)}$$

$$= Ku_1 - Ku_2$$

$$\therefore F_1 = K(u_1 - u_2) \quad \text{—— ①}$$

When only force F_2 is act in the element than.

$$K = \frac{F_2}{(u_2 - u_1)}$$

$$= -Ku_1 + Ku_2$$

$$\therefore F_2 = K(u_2 - u_1) \quad \text{—— ②}$$

Matrix eqn of ① & ①.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}_{2\times2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1}$$

$$F^{(e)}_{2\times1} = K^{(e)}_{2\times2} \; u^{(e)}_{2\times1} \quad \text{—— ③}$$

Where

$$K^{(e)} = K \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2}$$

$K^{(e)}$ → Element Stiffness matrix.

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Case 1 :→ When the body is subjected to axial load (tensile, compression) than element stiffness matrix is

$$K^{(e)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2}$$

A/C to Hookes law

$$\delta = E\epsilon$$

$$\frac{F}{A} = E\cdot\frac{\delta}{L}$$

$$K = \frac{F}{\delta} = \frac{AE}{L}$$

Case 2:→ When the body is subjected to torsion. Than element stiff. matrix is given by.

$$K^{(e)} = \frac{G\cdot J}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2}$$

A/c to torsion eqn

$$\frac{T}{J} = \frac{G f_s}{r} = \frac{G\theta}{L}$$

Long Book

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$$K = \frac{T}{\theta} = \frac{G\cdot J}{L}$$

Consider a body having stiffness 'K' s.t an internal force 'F' with linear deflection as 'u'

strain energy is given by

$$SE = \frac{1}{2}Ku^2 \quad \text{——} \ (1)$$

Potential energy due to external force is given by

$$PE = -F\cdot u \quad \text{——} \ (2)$$

Hence the total energy

$$T\cdot E = \frac{1}{2}Ku^2 - F u \quad \text{——} \ (3)$$

Long Book

For total Energy to min. differentiating eq^n ③ w.r.to u & equate to zero

$$\frac{d(TE)}{du} = 0 \quad (\text{minimization})$$

$$\frac{1}{2} K \cdot 2u - F = 0$$

$$\boxed{F = K \cdot u} \quad \text{—— ④}$$

Where eq^n ④ is the equilibrium eq^n for the analysis of element. But the component may have no. of elements. Hence eq^n ④ is written as

$$[F] = [K] \cdot [u] \quad \text{—— ⑤}$$

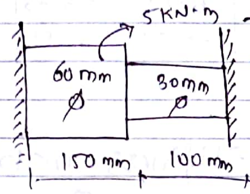
Where
F → Global force matrix.
K → Global stiffness matrix.
u → Global displ. matrix.

The displacement & force matrices are the coloum matrices whereas the stiffness matrix is the square matrix.

m → mm → ×10^3

Q] Determine the angle of twist at the step, the max^m shear stress in each section and the reaction at the wall for step circular shaft as sh. in fig. (14m W14)

$5 KN\cdot m \rightarrow 5\times10^3\times10^3 = 5\times10^6 N\cdot mm$

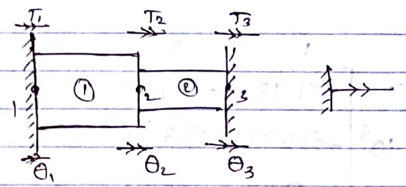


$G = 77\times10^9$ pa (N/m^2)

$= \frac{77\times10^9}{(1000)^2}$

$= 77\times10^3$ mpa

⇒ 1) Discritization



where,
T_1, T_2 & T_3 are the nodal torques (N-mm)
θ_1, θ_2 & θ_3 angular defl^n (radian)

② Element stiffness matrix

$$K^{(e)} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(G → N/mm^2, J → mm^4, L → mm)

$$K^{(1)} = \frac{77\times10^3 \times \frac{\pi}{32}\times(60)^4}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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① $K^{(1)} = 10^6 \begin{bmatrix} 651.93 & -651.93 \\ -651.93 & 651.93 \end{bmatrix}$ (DOF 1, 2)

$K^{(2)} = 10^6 \begin{bmatrix} 61.23 & -61.23 \\ -61.23 & 61.23 \end{bmatrix}$ (DOF 2, 3)

③ Global stiffness matrix

$K = K^{(1)} + K^{(2)}$

$K = 10^6 \begin{bmatrix} 651.93 & -651.93 & 0 \\ -651.93 & 713.16 & -61.23 \\ 0 & -61.23 & 61.23 \end{bmatrix}_{3\times3}$ (DOF 1, 2, 3)

mirror image

④ Nodal displacement

A/c to PMPE

$[T]_{3\times1} = [K]_{3\times3} \cdot [\theta]_{3\times1} \longrightarrow *$

Using Boundary Condition

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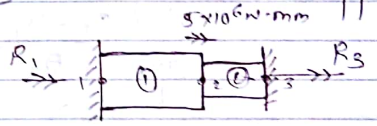
$10^6 \begin{bmatrix} 0 \\ +5 \\ 0 \end{bmatrix} = 10^6 \begin{bmatrix} & K & \end{bmatrix} \begin{bmatrix} 0 \\ \theta_2 \\ 0 \end{bmatrix}$ (DOF 1, 2, 3)

Using Gauss Elimination method.

$5 = 713.16 \times \theta_2$

$\theta_2 = 7.01 \times 10^{-3}$ rad

⑤ Reaction at support.



$R_1 = [1^{st}$ row of 'K' matrix$]_{1\times3} \times [\theta]_{3\times1}$

$= 10^6 \begin{bmatrix} 651.93 & -651.93 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 7.01\times10^{-3} \\ 0 \end{bmatrix}$

$R_1 = -4.57 \times 10^6$ N-mm

$= -4.57$ kN-m

$R_3 = \begin{bmatrix} 0 & -61.23 & 61.23 \end{bmatrix} \times \begin{bmatrix} 0 \\ 7.01 \\ 0 \end{bmatrix} \times 10^{-3}$

$R_3 = -429.42 \times 10^3$ N-mm

$= -0.429$ kN-m

DATE: mon^m

⑥ Stress in element.

$$f_s = \frac{T \cdot r}{J}$$

$$= \frac{T}{J} \cdot \frac{d}{2}$$

$$= \frac{T}{\frac{\pi}{32} d^4} \times \frac{d}{2}$$

$$f_s^{①} = \frac{16 T^{①}}{\pi d_1^3}$$

$$T^{①} = K \cdot \theta$$

$$= K [\theta_2 - \theta_1]$$

$$= \frac{G \cdot J_1}{L_1} [\theta_2 - \theta_1]$$

$$T^{①} = 4.57 \times 10^6$$

$$T^{②} = \frac{G \cdot J_2}{L_2} (\theta_3 - \theta_2)$$

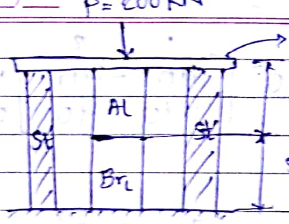
$$T^{②} =$$

$$f_s^{①} = 107.75 \ N/mm^2$$

$$f_s^{②} = -80.73 \ N/mm^2$$

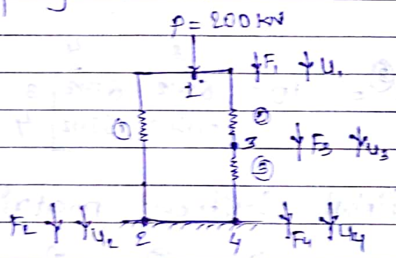
for 1-D element Pb. → x +ve, ← x −ve
for combined loading. y +ve ↓, ↑ y −ve

Q.B (Q.2 i)



⇒ 1] Discritization

Discritize the body into 3- 1D F.E & represent it in the form of spring element as sh. in figure.



where
$F_1 F_2 F_3 F_4$ be the nodal forces & $u_1 u_2 u_3 u_4$ be the nodal displ. at that node.

2] Element stiffness matrix

$$K^{②} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2} \ N/mm$$

$$= \frac{200 \times 2 \times 10^5}{1000} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 40 & -40 \\ -40 & 40 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$
(columns 1, 2)

$$K^{(2)} = \frac{370 \times 7 \times 10^4}{500} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 51.8 & -51.8 \\ -51.8 & 51.8 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix}$$
(columns 1, 3)

$$K^{(3)} = 10^3 \begin{bmatrix} 65.12 & -65.12 \\ -65.12 & 65.12 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$
(columns 3, 4)

3) Global Stiffness matrix

$$K = K^{(1)} + K^{(2)} + K^{(3)}$$

(columns 1, 2, 3, 4)

$$K = 10^3 \begin{bmatrix} 91.8 & -40 & -51.8 & 0 \\ -40 & 40 & 0 & 0 \\ -51.8 & 0 & 116.92 & -65.12 \\ 0 & 0 & -65.12 & 65.12 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

4) Nodal displacement

A/c to PMPE

$$[F] = [K][u]$$
$$4\times1 \quad 4\times4 \quad 4\times1$$

using Boundary condⁿ

$$10^3 \begin{bmatrix} 200 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} 91.8 & & -51.8 & \\ & & & \\ -51.8 & & 116.92 & \\ & & & \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} u_1 \\ 0 \\ u_3 \\ 0 \end{bmatrix}$$

(matrix labelled K, columns 1, 2, 3, 4)

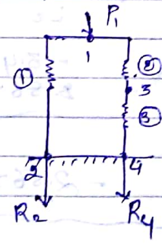
using gauss elimination

$$91.8\,u_1 - 51.8\,u_3 = 200$$
$$-51.8\,u_1 + 116.92\,u_3 = 0$$

$$u_1 = 2.98 \text{ mm}$$
$$u_3 = 1.28 \text{ mm}$$

Displ. matrix: $u = \begin{bmatrix} 2.98 \\ 0 \\ 1.28 \\ 0 \end{bmatrix}$ mm

5) Reaction at support.



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$R_2 = [\,2^{nd}$ row of K mat.$] \times [Q]$

$= [-40 \;\; 40 \;\; 0 \;\; 0] \begin{bmatrix} 2.98 \\ 0 \\ 1.28 \\ 0 \end{bmatrix}$

$R_2 = -119.2$ KN (↑)

directly

$R_4 = (P_1 - R_2) = -80.8$ KN (↑)

6) Max^m stresses & strain in element.

Strain $\epsilon^{(e)} = \frac{\delta L}{L}$

$\epsilon^{(1)} = \frac{(u_2 - u_1)}{L_1} = \frac{0 - 2.98}{1000}$

$\epsilon^{(1)} = -2.98 \times 10^{-3}$

$\epsilon^{(2)} = \frac{(u_3 - u_2)}{L_2} = \frac{1.28 - 2.98}{500}$

$\epsilon^{(2)} = -3.4 \times 10^{-3}$ (~~2.56~~)

$\epsilon^{(3)} = -2.56 \times 10^{-3}$

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Hooke's law $\sigma^{(e)} = E \cdot \epsilon^{(e)}$

$\sigma^{(1)} = E_1 \cdot \epsilon^{(1)}$

$= 2 \times 10^5 \times -2.98 \times 10^{-3}$ N/mm^2

$\sigma^{(1)} = -596$ N/mm^2 (Compression)

$\sigma^{(2)} = E_2 \cdot \epsilon_2$

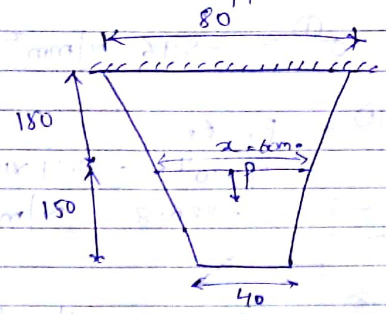
$= 7 \times 10^4 \times -3.4 \times 10^{-3}$

$\sigma^{(2)} = -226.8$ N/mm^2 (compr).

$\sigma^{(3)} = -225.28$ N/mm^2 (C)

Long Book

Q) For taper plate of uniform thickness t = 10 mm as sh. in fig & displ. at the nodes by forming into two element model. The plate material has mass density $\rho = 7800\ kg/m^3$, E = $2 \times 10^5\ N/mm^2$. In addition to self wt. the plate is s. to point load P = 1000 N at its centre. Also determine the reaction force at the support.

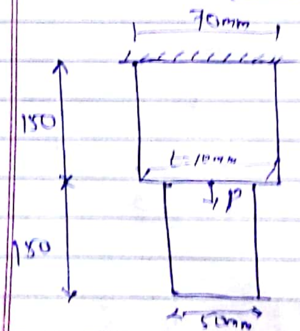


1) Discritization.

By similar Δ

$$\frac{80-40}{300} = \frac{80-x}{150}$$

$$x = 60mm$$



$A_1 = 70 \times 10 = 700\ mm^2$

$A_2 = 50 \times 10 = 500\ mm^2$

② Element stiffness matrix

$$K^{(e)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times2}$$

$$K^{(1)} = \frac{70 \times 10 \times 2 \times 10^5}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^5 \begin{bmatrix} 9.33 & -9.33 \\ -9.33 & 9.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$K^{(2)} = \frac{50 \times 10 \times 2 \times 10^5}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 6.6\ 10^5 \begin{bmatrix} 6.67 & -6.67 \\ -6.67 & 6.67 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

③ Global stiff. matrix.

$K = K^{(1)} + K^{(2)}$

$$K = 10^5 \begin{bmatrix} 9.33 & -9.33 & 0 \\ -9.33 & 16 & -6.67 \\ 0 & -6.67 & 6.67 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

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(4) Global load matrix.

a) Global self wt. matrix.

$w^{(1)} = \rho A L$ (N/mm³, mm², mm)

$\rho = 7800$ kg/m³ $= \frac{7800 \times 10 \text{ N}}{(1000)^3 \text{ mm}^3} = 78 \times 10^{-6}$ N/mm³

$= 78 \times 10^{-6} \times 700 \times 150$

$w^{(1)} = 8.19$ N

Distributing self wt. equally on two nodes.

$w^{(1)} = \begin{bmatrix} 4.095 \\ 4.095 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$

$w^{(2)} = \rho A L$

$= 78 \times 10^{-6} \times 500 \times 10 \times 150$

$w^{(2)} = 5.85$ N

$w^{(2)} = \begin{bmatrix} 2.925 \\ 2.925 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$

Global self wt. matrix.

$W = \begin{bmatrix} 4.095 \\ 7.03 \\ 2.93 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ 'N'

Long Book

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b) Global point load matrix

$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0 \\ 1000 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ (DOF)

Hence global load matrix is given by.

$P = W + F$

$= \begin{bmatrix} 4.095 \\ 1007.03 \\ 2.93 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ 'N'

(5) Nodal displacement

As per PMPE,

$[P]_{3\times1} = [K]_{3\times3} \cdot [u]_{3\times1}$

$\begin{bmatrix} 4.095 \\ 1007.03 \\ 2.93 \end{bmatrix} = 10^5 \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$

$1007.03 = 10^5 (16 u_2 - 6.67 u_3)$ — (1)

$2.93 = 10^5 (-6.67 u_2 + 6.67 u_3)$ — (2)

$u_2 = 1.075 \times 10^{-3}$ mm

$u_3 = 1.075 \times 10^{-3}$ mm.

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Ans displacement matrix

$$u = \begin{bmatrix} 0 \\ 0.0017 \\ 0.0018 \end{bmatrix} mm$$

(6) Reaction at support.

$$R_1 = [\,9.33 \;\; -9.33 \;\; 0\,] \begin{bmatrix} 0 \\ 0.0017 \\ 0.0018 \end{bmatrix}$$

~~$R_1 = 0.0158$~~

$R_1 = -1586.1\,(N)\ (\uparrow)$

(7) Stresses.

$$\sigma^{(1)} = \frac{E}{L}[u_2 - u_1]$$

$\sigma^{(1)} = 2.26\ N/mm^2\ (T)$

$\sigma^{(2)} = 0.133\ N/mm^2\ (T)$

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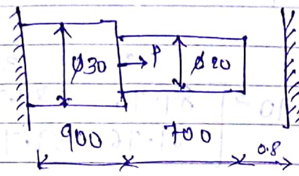
Q) Using 1-D FEM determine
1) displ. field
2) Stress & strain
3) Reactn at support.

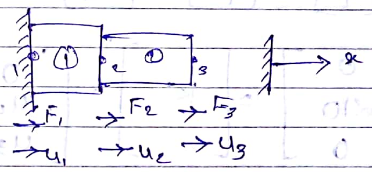


$P = 210\ kN$

$E = 2\times10^5\ mpa$

→ In this pb. we should first determine whether contact betn bar & fixed wall occurs or not. To do this we should 1st determine the displ. of a bar at its free end. Hence consider a bare as a cantilever beam.

1) Discritization



(2) Element stiffness matrix

$$K^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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$= \frac{706.85 \times 2\times10^5}{900}$

$= 10^3 \begin{bmatrix} 157.08 & -157.08 \\ -157.08 & 157.08 \end{bmatrix}$ (nodes 1, 2)

$K^{(2)} = 10^3 \begin{bmatrix} 89.76 & -89.76 \\ -89.76 & 89.76 \end{bmatrix}$ (nodes 2, 3)

3) Global stiffness matrix.

$K = K^{(1)} + K^{(2)}$

$K = 10^3 \begin{bmatrix} 157.08 & -157.08 & 0 \\ -157.08 & 246.84 & -89.76 \\ 0 & -89.76 & 89.76 \end{bmatrix}$

4) Nodal displ. mat.

$[F] = [K] \cdot [u]$

$10^3 \begin{bmatrix} 0 \\ 210 \\ 0 \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$

$210 \times 10^3 = 246.84\, u_2 - 89.76\, u_3$

$0 = -89.76\, u_2 + 89.76\, u_3$

$u_2 = 1.33$ mm

$u_3 = 1.33$ mm

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Where u_3 be the displacement at free end of the bar. From this result it clears that the contact betn bar & fixed wall does occurs. Hence changing the boundary condn of the displ. & recal. the displ. u_2.

Now $u_1 = 0$, $u_2 = ?$, $u_3 = 0.8$.

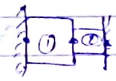
$210 = 246.84\, u_2 - 89.76 \times 0.8$

$0 = -89.76\, u_2 + 89.76 \times 0.8$

$u_2 = 1.14$ mm

Ans. Displ. matrix. $u = \begin{bmatrix} 0 \\ 1.14 \\ 0.8 \end{bmatrix}$

5) Reaction at support.



$R_1 = 10^3 \begin{bmatrix} 157.08 & -157.08 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1.14 \\ 0.8 \end{bmatrix}$

$R_1 = -179.07$ KN (←)

$R_3 = -30.51$ KN (←)

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6) Stresses & Strain.

Strain: $\epsilon^{(1)} = \frac{\delta L}{L}$

$$\epsilon^{(1)} = \frac{u_2 - u_1}{L_1}$$

$$= \frac{1.14 - 0}{900}$$

$$= 1.26 \times 10^{-3}$$

$$\epsilon^{(2)} = \frac{u_3 - u_2}{L_2}$$

$$\epsilon^{(2)} = -4.85 \times 10^{-4}$$

Stresses:

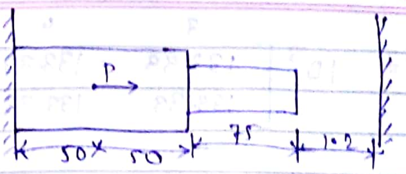
$$\sigma^{(1)} = \frac{E}{L}[u_2 - u_1]$$

$$= \frac{2\times10^5}{900}[1.14 - 0]$$

$$\sigma^{(1)} = 253.33 \text{ mpa (T)}$$

$$\sigma^{(2)} = -97.14 \text{ mpa (C)}$$

Q.25



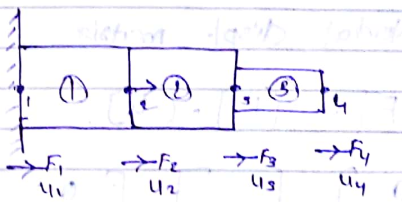
$E = 2\times10^5$ mpa

$A_1 = 100\ mm^2$

$A_2 = 50\ mm^2$

$P = 100$ N

1) Discritization.



2) Element stiff. mat.

$$K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{(1)} = 10^3 \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$K^{(2)} = 10^3 \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

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$$K^{(3)} = 10^3 \begin{bmatrix} 133.33 & -133.33 \\ -133.33 & 133.33 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

(columns 3, 4)

(3) Global Stiff. matrix.

$$K = 10^3 \begin{bmatrix} 400 & -400 & 0 & 0 \\ -400 & 800 & -400 & 0 \\ 0 & -400 & 533.33 & -133.33 \\ 0 & 0 & -133.33 & 133.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(columns 1, 2, 3, 4)

(4) Nodal displ. matrix.

$[F] = [K] \cdot [u]$.

$$\begin{bmatrix} 0 \\ 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

~~200×10^3~~ $100 = 800\times10^3 u_2 - 400\times10^3 u_3$

$0 = -400\times10^3 u_2 + 533.33 u_3 - 133.33 u_4$

$0 = -133.33 u_3 + 133.33 u_4$

$u_2 = 5\times10^{-4}$

$u_3 = 5\times10^{-4}$

$u_4 = 5\times10^{-4}$

$133.33 u_3 = 133.33 u_4$.

$u_3 = u_4$

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Q.B Thermal

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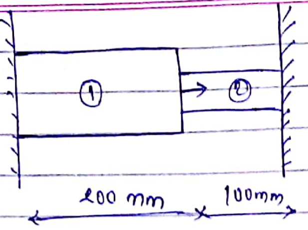
Q4] b]



$P = 350$ kN

$E_1 = 70$ GPa

$A_1 = 1200\ mm^2$

$\alpha_1 = 23 \times 10^{-6}/°C$

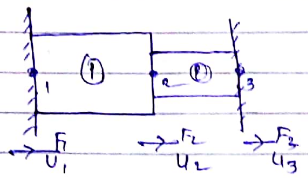
$E_2 = 200$ GPa

$A_2 = 800\ mm^2$

$\alpha_2 = 11.7 \times 10^{-6}/°C$

$\Delta T = 30°C$

1) Discritization



2) Element Stiff. mat.

$$K^{(1)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 420 & -420 \\ -420 & 420 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(columns 1, 2)

$$K^{(2)} = 10^3 \begin{bmatrix} 1600 & -1600 \\ -1600 & 1600 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

(columns 2, 3)

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3) Global Stiff. mat.

$$K = 10^3 \begin{bmatrix} 420 & -420 & 0 \\ -420 & 2020 & -1600 \\ 0 & -1600 & 1600 \end{bmatrix}_{3\times3} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

(columns 1, 2, 3 ← DOF)

4) Global load matrix

a) Global thermal force matrix

$$\theta^e = EA\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times1}$$

$$\theta^{(1)} = 10^3 \begin{bmatrix} -57.96 \\ 57.96 \end{bmatrix}_{2\times1} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\theta^{2} = 10^3 \begin{bmatrix} -56.16 \\ 56.16 \end{bmatrix}_{2\times1} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$\theta = 10^3 \begin{bmatrix} -57.96 \\ 1.8 \\ 56.16 \end{bmatrix}_{3\times1} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

(↓ DOF)

b] Global pt./nodal force matrix

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 350 \\ 0 \end{bmatrix} \times 10^3 \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

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∵ P = Global load matrix is given by

$P = \theta + F$

$$\{P\} = 10^3 \begin{bmatrix} -57.96 \\ 351.8 \\ 56.16 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \text{DOF}$$

5) Nodal displ. mat. / displ. field.

A/c to PMPE

$P_{3\times1} = K_{3\times3} \times u_{3\times1}$

Using Boundary Condn.

$$10^3 \begin{bmatrix} -57.96 \\ 351.8 \\ 56.16 \end{bmatrix} = 10^3 \begin{bmatrix} & k & \end{bmatrix} \times \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix}$$

$351.8 = 2020\, u_2$

$u_2 = 0.174$ mm

Ans displ. field, $u = \begin{bmatrix} 0 \\ 0.174 \\ 0 \end{bmatrix}_{3\times1}$ mm (DOF 1, 2, 3)

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6) Reactn at support.

$R_1 = \begin{bmatrix} 420 & -420 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0.174 \\ 0 \end{bmatrix}$

$R_1 = -73.08$ KN (←)

$R_3 = \begin{bmatrix} 0 & -1600 & 1600 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0.174 \\ 0 \end{bmatrix}$

$R_3 = -278.4$ KN (←)

7) Stresses

$\sigma^{(1)} = \frac{E_1}{L_1}(u_2 - u_1) - E_1 \alpha \Delta T$

$\sigma^{(1)} = 12.6$ mpa (T)

$\sigma^{(2)} = -418.2$ mpa (C)

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Thermal

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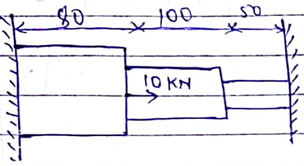
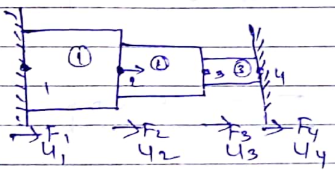
W-14 Q] 13 m For the 3 step bar sh. in figure the bar fit snugly betn the rigid wall at room temp. The temp is then raised by 40°C. Det displ. & stresses.



$E_1 = 70$ Gpa
$A_1 = 1000\ mm^2$
$\alpha_1 = 23 \times 10^{-6} / °C$
$E_2 = 105$ Gpa
$A_2 = 500\ mm^2$
$\alpha_2 = 19 \times 10^{-6} / °C$
$E_3 = 210$ Gpa
$A_3 = 250\ mm^2$
$\alpha_3 = 12 \times 10^{-6} / °C$

⇒ 1) Discritization.



2) Element stiffness matrix.

$$K^{(1)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 875 & -875 \\ -875 & 875 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$
(columns 1, 2)

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$$K^{(2)} = 10^3 \begin{bmatrix} 525 & -525 \\ -525 & 525 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$
(columns 2, 3)

$$K^{(3)} = 10^3 \begin{bmatrix} 1050 & -1050 \\ -1050 & 1050 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$
(columns 3, 4)

3) Global stiff. mat.

$$K = \begin{bmatrix} 875 & -875 & 0 & 0 \\ -875 & 1400 & -525 & 0 \\ 0 & -525 & 1575 & -1050 \\ 0 & 0 & -1050 & 1050 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$
(columns 1, 2, 3, 4)

4) Global load matrix

$$\theta^{(1)} = EA\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\theta^{(1)} = 10^3 \begin{bmatrix} -64.4 \\ 64.4 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\theta^{(2)} = 10^3 \begin{bmatrix} -39.9 \\ 39.9 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$\theta^{(3)} = 10^3 \begin{bmatrix} -25.2 \\ 25.2 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

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$$\theta = \begin{bmatrix} -64.4 \\ 24.5 \\ 14.7 \\ 25.2 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

b] Global pt. mat.

$$F = 10^3 \begin{bmatrix} 0 \\ 10 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$\therefore$ $P = F + \theta$

$$= 10^3 \begin{bmatrix} -64.4 \\ 34.5 \\ 14.7 \\ 25.2 \end{bmatrix}$$

5) Nodal displ. matrix

$P = K \times U$

$$10^3 \begin{bmatrix} -64.4 \\ 34.5 \\ 14.7 \\ 25.2 \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \times \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{bmatrix}$$

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$$34.5 = \begin{bmatrix} -875 & 1400 & -525 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{bmatrix}$$

$34.5 = 1400 u_2 - 525 u_3$

$14.7 = -525 u_2 + 1575 u_3$

~~$25.2 = 0 + 0 u_2 + 1050 u_3$~~

$u_2 = 0.032$

$u_3 = 0.020$

Ans displ. mat. $U = \begin{bmatrix} 0 \\ 0.032 \\ 0.020 \\ 0 \end{bmatrix}$

6) Reaction @ support.

$R_1 = -28$ KN $(\leftarrow)$

$R_4 = -21$ KN $(\leftarrow)$

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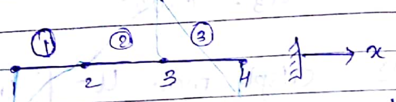
Q] A finite element sol^{n} using 1-D 2 noded element has been obtained for the rod as sh. in fig. displ. are as follows.

$$u = [-0.2,\ 0,\ 0.6,\ -0.1]^T \text{ mm}$$

$E = 1 N/mm^2$

$A = 1 mm^2$

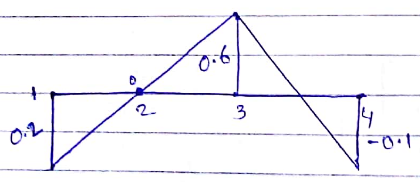
$L_{1-2} = 50mm$, $L_{2-3} = 80mm$, $L_{3-4} = 100 mm$



A/c to finite element theory.

1) Plot displ. $u(x)$ Vs x
2) Plot $\epsilon(x)$ Vs x
3) Determine strain displ. relation mat 'B' for element 2-3
4) Determine strain energy in the element 1-2. $\left(\frac{1}{2} u^T \cdot K \cdot u\right)$

Sol^{n} ⇒ 1) Plot displ. $u(x)$ Vs x



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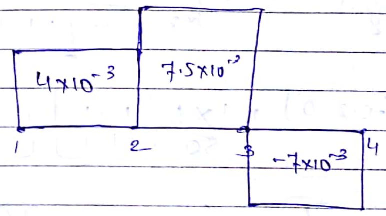
2) Plot $\epsilon(x)$ Vs x

$$\epsilon^{(1)} = \frac{u_2 - u_1}{L_{1-2}} = 4 \times 10^{-3}$$

$$\epsilon^{(2)} = \frac{u_3 - u_2}{L_{2-3}} = 7.5 \times 10^{-3}$$

$$\epsilon^{(3)} = \frac{u_4 - u_3}{L_{3-4}} = -7 \times 10^{-3}$$



3) Strain displ. relation mat. 'B' for 2-3

$$\epsilon^{(e)} = \frac{(u_2 - u_1)}{L}$$

$$\epsilon^{(1)}_{1\times1} = \frac{1}{L}[-1 \quad 1]_{1\times2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1}$$

$$\epsilon^{(1)}_{1\times1} = B_{1\times2} \cdot u_{2\times1}$$

$$B_{2-3} = \frac{1}{L_{2-3}}[-1 \quad 1]$$

$$B_{2-3} = \frac{1}{80}\begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times 2}$$

4] Strain energy in element

$$S.E_{1-2} = \frac{1}{2} u^T . K . u$$

$$= \frac{1}{2}\begin{bmatrix} -0.2 & 0 \end{bmatrix}_{1\times 2} \times \frac{AE}{L}\begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2}\begin{bmatrix} -0.2 \\ 0 \end{bmatrix}_{2\times 1}$$

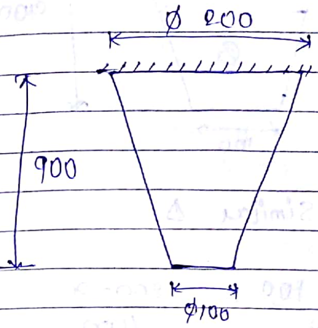
$$= \frac{1}{2}\begin{bmatrix} -0.2 & 0 \end{bmatrix} \times \frac{1\times 1}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -0.2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1 & 0 \end{bmatrix} \times \begin{bmatrix} 0.02 & -0.02 \\ -0.02 & 0.02 \end{bmatrix}\begin{bmatrix} -0.2 \\ 0 \end{bmatrix}$$

$$SE_{1-2} = 4 \times 10^{-4}\ N\text{-}mm$$

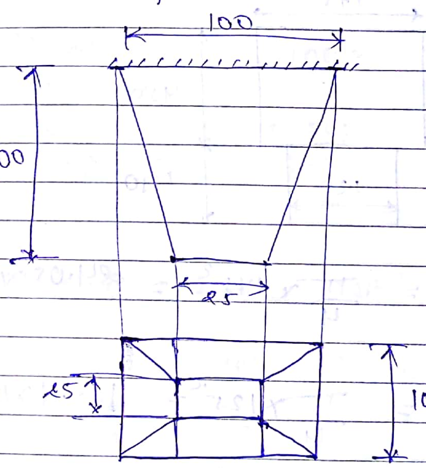
$$S.E_{1-2} = 4 \times 10^{-7}\ N.m\ (J)$$

Q)1 Cal. a displ. of a bar at its free end under its self wt



Take
$E = 100\ Gpa$
$\rho = 15 \times 10^{-9}\ N/mm^3$

Q]2 find the displ. of a taper bar at its free end under its own wt



Take
$E = 200\ Gpa$
$\rho = 77 \times 10^{-6}\ N/mm^3$

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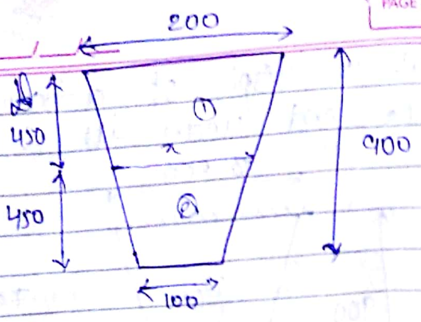
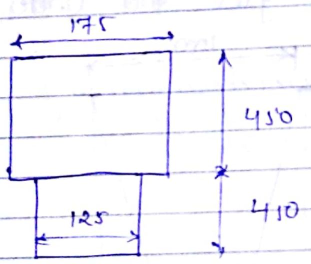
Q1) 1)



By Similar Δ

$$\frac{200-100}{900} = \frac{200-x}{450}$$

$$x = 150 \text{ mm}$$



$$A_1 = \frac{\pi}{4} \times 175^2 = 24.05 \times 10^3 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} \times 125^2 = 12.27 \times 10^3 \text{ mm}^2$$

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2) Element Stiff. matrix.

$$K^{(1)} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{24.05 \times 10^3 \times 100 \times 10^3}{450} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 5345 & -5345 \\ -5345 & 5345 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(columns 1, 2)

$$K^{(2)} = 10^3 \begin{bmatrix} 2727 & -2727 \\ -2727 & 2727 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

(columns 2, 3)

3) Global Stiff. matr.

$$K = 10^3 \begin{bmatrix} 5345 & -5345 & 0 \\ -5345 & 8072 & -2727 \\ 0 & -2727 & 2727 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

(columns 1, 2, 3)

4) Global load matrix

a) Global self wt. mat

$$w^{(1)} = \rho A_1 L$$

$$= 15 \times 10^{-6} \times 24.05 \times 10^3 \times 450$$

$w^{(1)} =$ ~~833.33 N~~

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Distributing load.

$$w^{(1)} = \begin{bmatrix} 416.66 \\ 416.66 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$w^{(2)} = \rho A_e L$$

$$= 75 \times 10^{-6} \times 12.27 \times 10^{3} \times 450$$

$$w^{(2)} = 425.15 \, N$$

$$w^{(2)} = \begin{bmatrix} 212.57 \\ 212.57 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Global self. wt. matrix

$$w = \begin{bmatrix} 416.66 \\ 629.23 \\ 212.57 \end{bmatrix}$$

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Langrages shape fn"

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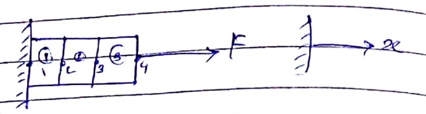
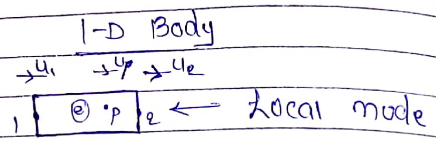
★ (9) Shape function for 1-D bar element or 1-D line element (2-noded linear shape function)

⇒





Consider a body divided into finite element as sh. in fig. Displacement of each node is possible only in x-dirn, hence only 1 parameter can be expressed displ. of each node. So each node have 1 DOF hence the element is kn. as 1-D linear bar / line element.

★ Shape function →

It is a fun or parameter or eqn, which define the displ. within an element in terms of nodal displ. is called as shape function.

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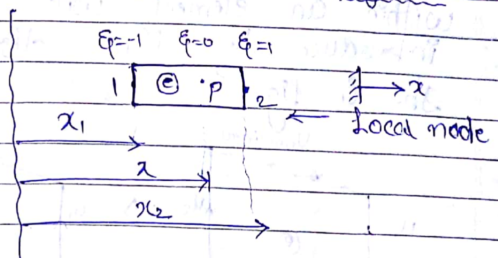
★ Natural Co-ordinate system :→



Let x_1, x_2 be the co-ordinate of node 1, 2 respt.

Hence we define the natural co-ordinate system & is denoted by ξ & is given by.

$$\xi = \left\{ \frac{2}{(x_2 - x_1)} \cdot (x - x_1) \right\} - 1 \quad \text{———} (1)$$

Use $x = x_1$, $\xi = -1$ for node 1.

Use $x = x_2$, $\xi = 1$ for node 2.

Use $x = \frac{x_1 + x_2}{2}$ (mid pt), $\xi = 0$

ξ varies from -1 to 1 i.e length of an element is covered when ξ changes from -1 to 1. Hence according to natural co-ordinate system the length of an element is +2.

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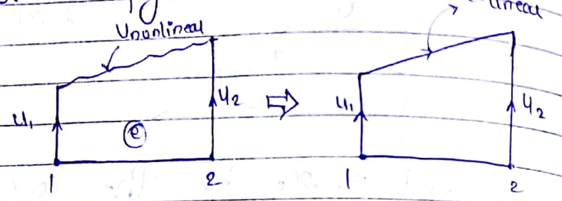
Now the unknown displ. field within an element will be introduce by linear distribution as sh. in fig.



The above approximation will be more accurate if more no. of element is considered in model.

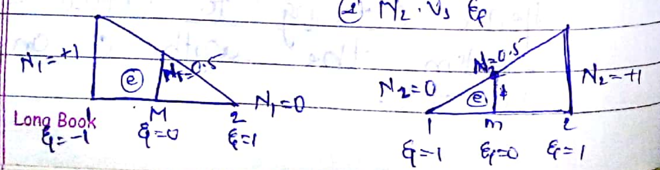
To implement the linear displ. linear shape fun will be introduced as,

$$N_1 = \frac{1-\xi}{2}, \quad N_2 = \frac{1+\xi}{2} \quad \text{—— (2)}$$

Where,
N_1 & N_2

Plot the graph bet shape fun & natural Co-ordinate.

(1) N_1 Vs ξ (2) N_2 Vs ξ



From the above graph the addition of above eqn at any pt. including nodes are equal to 1. i.e

$$N_1 + N_2 = 1 \quad \text{—— (3)}$$

Once the shape fun are defined the linear displ. field within an element can be written in terms of nodal displ. u_1 & u_2 as

$$u_p = N_1 u_1 + N_2 u_2 \quad \text{—— (4)}$$

Where,
u_p → displ at any pt. 'P' within an element.

Similarly we can write

$$x_p = N_1 x_1 + N_2 x_2 \quad \text{—— (5)}$$

x_p → Co-ordinate at any pt. P within an element.

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★ Significance of Shape Function

1) No. of the shape fun are equal to no. of nodes in the element.
2) The eqn of shape fun represented in terms of natural Coordinate.

3) Addn of shape fun are equal to 1 at any pt. including nodes

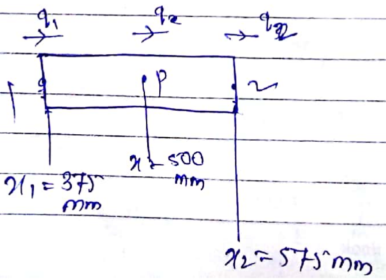
Q1] Consider a bar sh. in fig.

$A = 625$ mm

$E = 200 \times 10^3$ mpa

If $q_1 = 0.5$ mm, $q_2 = 0.625$ mm

1) derive the relationship & determine disp. at pt 'P'.
2) Strain ϵ & stress σ
3) Element Stiff. mat.
4) Strain energy in the element.



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1) $q_p = N_1 q_1 + N_2 q_2$ —— (1)

$N_1 = \frac{1-\xi}{2}$

$N_2 = \frac{1+\xi}{2}$ —— (2)

$\xi_p = \left\{ \frac{2}{(x_2 - x_1)} (x - x_1) \right\} - 1$

$= \left\{ \frac{2}{(575 - 375)} (500 - 375) \right\} - 1$

$\xi_p = 0.25$

$N_1 = \frac{1 - 0.25}{2} = 0.375$

$N_2 = \frac{1 + 0.25}{2} = 0.625$

$q_p = 0.375 \times 0.5 + 0.625 \times 0.625$

$q_p = 0.5781$

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2) $\epsilon^{(e)} = \dfrac{\text{Change in length}}{\text{Original length}}$

$= \dfrac{(q_2 - q_1)}{(x_2 - x_1)} =$

$= \dfrac{(0.625 - 0.375)}{575 - 375}$

$= \underline{6.25 \times 10^{-4}}$

A/c Hooke's Law.

$\sigma^{(e)} = E \cdot \epsilon$

$\sigma = = 2 \times 10^5 \times 6.25 \times 10^{-4}$

$\sigma^{e} = 125 \ N/mm^2$

3) Element stiff. matrix

$K^{(e)} = \dfrac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad (x_2 - x_1)$

$= \dfrac{625 \times 2 \times 10^5}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= 10^3 \begin{bmatrix} 625 & -625 \\ -625 & 625 \end{bmatrix} N/mm$

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4] Strain Energy.

$SE = \dfrac{1}{2} F_{2\times1} \cdot u_{2\times1}$

$= \dfrac{1}{2} K_{2\times2} \cdot u_{2\times1} \cdot u_{2\times1}$

$= \dfrac{1}{2} u^T_{1\times2} \cdot K_{2\times2} \cdot u_{2\times1}$

$= \dfrac{1}{2} [0.5 \ \ 0.625]_{1\times2} \ 10^3 \begin{bmatrix} 625 & -625 \\ -625 & 625 \end{bmatrix}_{2\times2} \begin{bmatrix} 0.5 \\ 0.625 \end{bmatrix}_{2\times1}$

$F^e = K \cdot u$

$K \cdot u = u^T \cdot K^T$ $\quad (2\times2 \ 2\times1 \quad 1\times2 \ 2\times2)$

$K \cdot u = u^T K$

$K^T = K$

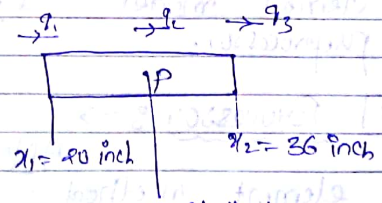
$S.E = 4882.81 \ N\text{-}mm$

$= \underline{4.88 \ N\text{-}m \ (J)}$

Q] Evaluate ξ, N_1 & N_2 at pt. 'P'

if $q_1 = 0.003$ inch, $q_2 = -0.005$ inch

determine the value of disp. 'q' at pt. 'P'



$\xi_1 = -0.5$

$N_1 = 0.75$

$N_2 = 0.75$

$q_P = -1.5 \times 10^{-3}$

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Q) STEPS Of FEM any finite Element analysis using any software package.

1) Preprocessor
2) Processor
3) Post-processor.

1] Preprocessor:→ Any body can be analysed using FEM first by discritizing the body into elements & nodes i.e mesh is generated. The loads acting on the body as well as the boundary condⁿ are also noted. This complete task is grouped under finite element model. The process upto finite element model called as preprocessor.

2] Processor :→ After finite element method element matricies are formed which are assemble together to get matrix eqn to get global. This eqn are solve to get the values of field variable at the nodel pt which is treated as solⁿ. There task after FEM is called as FEA and the processes followed upto FEA after FEM is called as processor.

3] Post processor :→ The result are analysed using a graphical tool for the modification of the design this process is called as postprocessor. The software which carried out task before FEA is called as preprocessor, while the software which carries out the task after FEA is called as post-processor.

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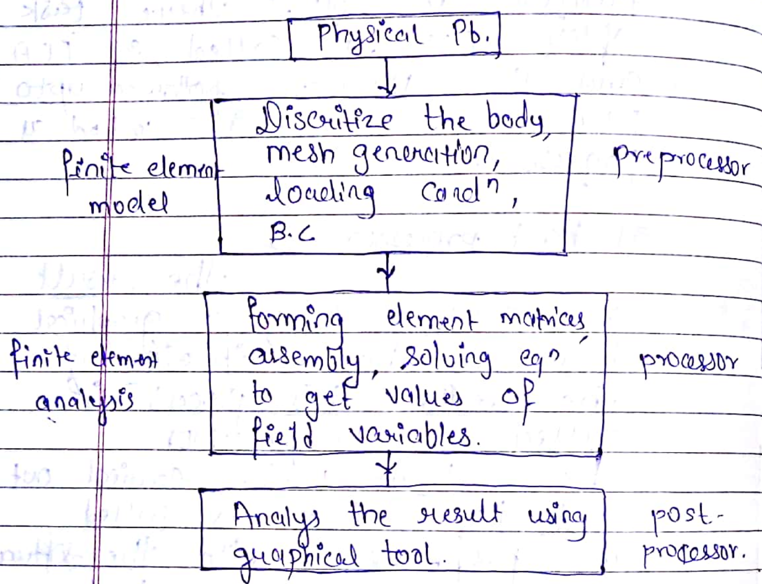
Q] Difference bet FEM & FEA

Q] Flow Chart.



Q] Types of Error in FEM

⇒ The FEA is an approximate numerical method to obtained the sol^n which is not exact because of foll. types of error

1) Mathematical modelling error
2) Discritization error
3) Round off error.

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1) Mathematical modelling error :→
This error is due to approximation of physical system & assumptions.

2) Discritization error :→
This error is due to the size, qty. & type of element to be considered in the model.

3) Round-off error :→
This error is due to accuracy level possible due to fixed no. of digits.

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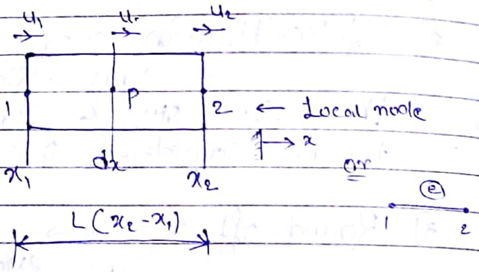
Q) Derive strain displ. relation matrix 'B' or Element stiffness matrix 'K' for 1-D bar or line element. (2-noded using shape fun approach) (St-(SF approach))



The strain at a point within an element is given by

$$\epsilon = \frac{\partial u}{\partial x} \quad \text{—— (1)}$$

By chain differentiation rule we can write.

$$\epsilon = \frac{\partial u}{\partial \epsilon_l} \cdot \frac{\partial \epsilon_l}{\partial x} \quad \text{—— (2)}$$

Where

$u \rightarrow$ displ. at any pt.

$$u = N_1 u_1 + N_2 u_2$$

$$N_1 = \frac{1-\epsilon_l}{2}, \quad N_2 = \frac{1+\epsilon_l}{2}$$

$$u = \left(\frac{1-\epsilon_l}{2}\right) u_1 + \left(\frac{1+\epsilon_l}{2}\right) u_2$$

Differentiate w.r.t. to ϵ_l (differentiate)

Considering $\frac{\partial u}{\partial \epsilon_l} = \frac{-u_1}{2} + \frac{u_2}{2}$

$$= \left(\frac{u_2 - u_1}{2}\right) \quad \text{—— (3)}$$

Where

$$\epsilon_l = \left\{\frac{2}{(x_2 - x_1)} \cdot (x - x_1)\right\} - 1$$

differentiate w.r.t. to x

Consider $\frac{\partial \epsilon_l}{\partial x} = \frac{2}{(x_2 - x_1)}$ —— (4)

using eqn (3) & (4) in (2)

$$(2) \Rightarrow \epsilon = \left(\frac{u_2 - u_1}{2}\right) \cdot \left(\frac{2}{x_2 - x_1}\right)$$

$$\epsilon_{1\times1} = \underbrace{\frac{1}{(x_2 - x_1)} \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times2}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1}$$

$$\epsilon_{1\times1} = B_{1\times2} \cdot u_{2\times1} \quad \text{—— (5)}$$

Where

$$B = \frac{1}{(x_2 - x_1)} \begin{pmatrix} -1 & 1 \end{pmatrix}_{1\times2}$$

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$S.E_{\text{at a pt.}} / \text{volume} = \frac{1}{2} \sigma^T . \epsilon$ —— (6)

For understanding:

$SE = \frac{1}{2} . F . \delta$

$\sigma = \frac{F}{A} \quad \therefore F = \sigma . A$

$\epsilon = \frac{\delta}{L} \quad \therefore \delta = \epsilon . L$

$SE = \frac{1}{2} . \sigma . A \times \epsilon . L$

$SE = \frac{1}{2} \sigma . \epsilon . V \quad \because V = A \times L$

$SE/V = 1/2 \, \sigma . \epsilon$

$SE \text{ at pt} = \frac{1}{2} \sigma^T . \epsilon \times \text{vol}$

$= \frac{1}{2} \times \sigma^T \times \epsilon \times A . dx$ —— (7)

To get the value of total S.E for 1-D bar element integrating eq^n (7)

$\text{Total SE} = \int_{x_1}^{x_2} 1/2 \; \sigma^T . \epsilon . A . dx$

$T.S.E = \frac{1}{2} \sigma^T . \epsilon . A . (x_2 - x_1)$

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$= \frac{1}{2} E . u^T . B^T . B . u . A \; (x_2 - x_1)$

$= \frac{1}{2} u^T A E \, (x_2 - x_1) \, B^T . B . u$

$T.S.E = \frac{1}{2} u^T . K . u$ —— (8)

Where, $K = AE \, (x_2 - x_1) \, B^T B$

$K = AE \, (x_2 - x_1) . \frac{1}{(x_2 - x_1)} \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times1} . \frac{1}{(x_2 - x_1)} \begin{bmatrix} -1 & 1 \end{bmatrix}_{1\times2}$

$K = \frac{AE}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\boxed{K = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

$\epsilon = B . u$

Hookes law.

$\sigma = E . \epsilon$

$\sigma = E \, B . u$

$\sigma^T = E \, (Bu)^T = E . u^T . B^T$

$SE = \frac{1}{2} F . u \qquad \because F = K . u$

$= \frac{1}{2} (K . u) . u \qquad Ku = (Ku)^T$

$= \frac{1}{2} (K . u)^T . u \qquad = u^T K^T$

$= \frac{1}{2} u^T K^T u \qquad K^T = K$

$= 1/2 \; u^T K . u$

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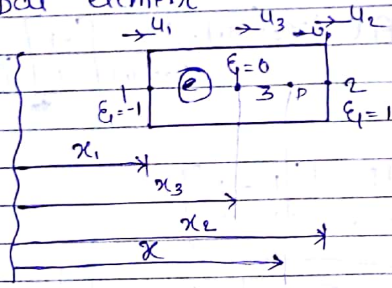
Shape fun for 1-D quadratic bar element / Higher order element.

Use → for more accurate rslt of stress analysis we will introduce a new 3 noded element as sh. in fig. kn. as 3-noded quadratic bar element



Consider a typical 3-noded quadratic bar element as sh. in fig.
In the local no. scheme, the left node will be node 1, the right node is 2 & the mid-pt node is 3.
Let x_1, x_2 & x_3 are the co-ordinates of node 1, 2 & 3 respt.
The natural co-ordinate system is defined for 1-D quadratic bar element

$$\xi = \frac{2(x - x_3)}{(x_2 - x_1)} \quad \text{--- (1)}$$

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use $x = x_1$, $\xi = -1$
$x = x_2$, $\xi = 1$
$x = x_3$, $\xi = 0$

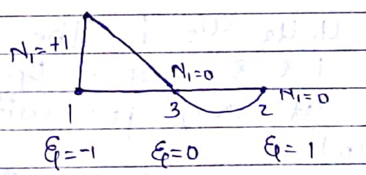
From this result we see that the natural co-ordinates $\xi = -1$, $\xi = 0$, $\xi = +1$ at node 1, 3 & 2 respt.

Let the quadratic shape fun N_1, N_2 & N_3 will be introduce as.

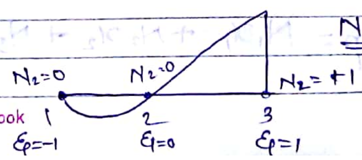
$$\left.\begin{aligned} N_1 &= \frac{-1}{2}\xi(1-\xi) \\ N_2 &= \frac{1}{2}\xi(1+\xi) \\ N_3 &= (1-\xi)(1+\xi) \end{aligned}\right\} \quad (2)$$

Now, plot the graph betn shape fun N_1, N_2 & N_3 Vs ξ

N_1 Vs ξ



N_2 Vs ξ



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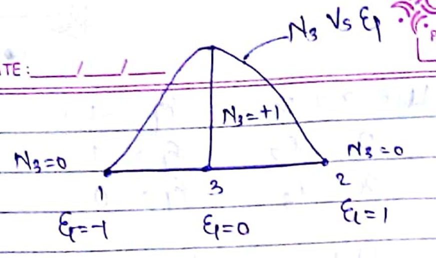

From the above plot it is clear that the addition of shape fn at any pt. including nodes are equal to +1

i.e $N_1+N_2+N_3 = +1$ ⟵ (3)

at node 1, $N_1=1$, $N_2=0$, $N_3=0$
at node 2, $N_2=1$, $N_3=N_1=0$
at node 3, $N_3=1$, $N_2=N_1=0$

One the shape fn are defined the unknown displ. field within an element will be interpolated by

$U_p = N_1U_1 + N_2U_2 + N_3U_3$ ⟵ (4)

Where U_1 U_2 & U_3 be the displ. at node 1 2 3 respt. U_p be the displ. at any pt. within an element.
Similarly we can write.

$x_p = N_1x_1 + N_2x_2 + N_3x_3$ ⟵ (5)

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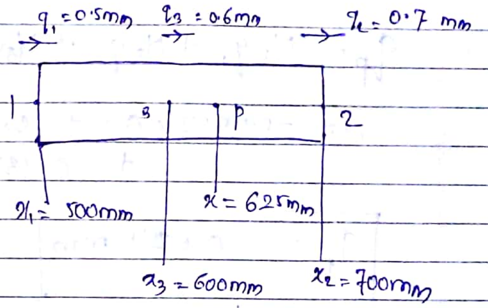
where x_1 x_2 x_3 be the ~~dis~~ co-ordinate at node 1 2 3 respt & x_p be the co-ordinate at pt 3 within an element.

Que]. Determine displ. at pt. P as sh. in fig. using shape fn approach.



⇒ The displ. @ pt. 'P' is given by

$q_p = N_1q_1 + N_2q_2 + N_3q_3$ ⟵ (1)

$\xi_p = \frac{2(x-x_3)}{(x_2-x_1)} = 0.25$

$N_1 = \frac{-1}{2}\xi(1-\xi)$

$= -0.0937$

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$N_2 = \frac{1}{2} \xi (1+\xi)$

$= 0.156$

$N_3 = (1-\xi)(1+\xi)$

$= 0.937$

$q_p = N_1 q_1 + N_2 q_2 + N_3 q_3$

$= -0.0937 \times 0.5 + 0.156 \times \sout{0.6}\ 0.7 + 0.937 \times 0.6$

$\boxed{q_p = 0.624 \text{ mm}}$

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Imp

Bandwidth of matrices / Global stiffness matrix

The max^m width of non zero element of matrix is called as bandwidth of a matrix

Significance

It should be kept min so as to take less space to store in a memory of the computer.

eg:-

$$K = \begin{bmatrix} a & b & 0 & 0 & 0 \\ b & c & d & 0 & 0 \\ 0 & d & e & f & 0 \\ 0 & 0 & f & g & h \\ 0 & 0 & 0 & h & i \end{bmatrix}_{5\times5} \quad B.W = 03$$

$$K_{banded} = \begin{bmatrix} b & a & b \\ d & c & d \\ f & e & f \\ h & g & h \\ 0 & i & 0 \end{bmatrix}_{5\times3}$$

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Actual stiffness matrix is denot. by 'K' & bandwidth of matrix = 3. To store it in the memory of computer in the form of banded matrix K'_{banded} in its diagonal form neglecting upper & lower triangular zero elements.

To minimize the bandwidth of global stiffness matrix the connectivity of element is as sh. below, for eg.

1) for 1-D or line element

① — ② — ③ - - - - - - - ⓘ
1 2   2 3   3 4   i i+1

2) CST element

① — ② — ③ - - - - ⓘ
1 2 3   2 3 4   3 4 5   i i+1 i+2

Unit 2

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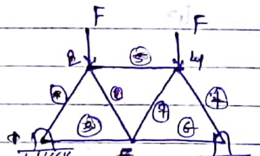
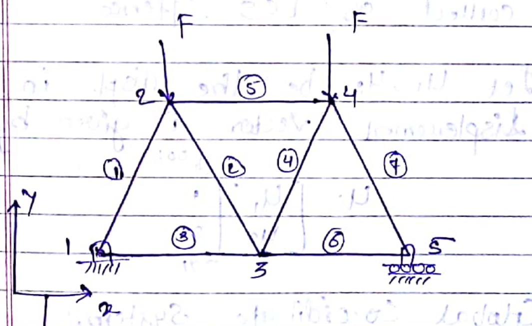
Imp Q) 6) Derive element stiffness matrix for 1-D line element in terms of Global co-ordinate system. (GCS)

Or

Derive Element stiffness matrix of a truss element.

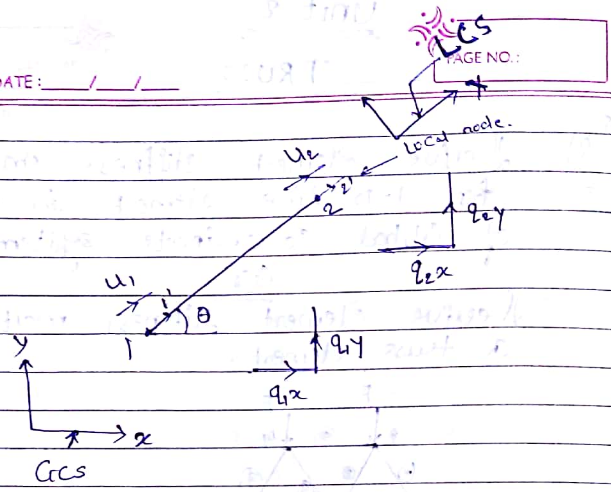


Discritize the truss into 7 1-D line elements considering each member as 1 element.



Global co-ordinate system (GCS)

Consider any ith element inclined at an angle "θ" in c.c.w. dirn as sh. in fig.



Local Co-ordinate system :-

It is the Co-ordinate system in which displ. of each node represent only in one dirn. called as LCS. Hence

Let u_1, u_2 be the displ. in LCS. Displacement vector is given by

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(DOF; 2×1)

Global Co-ordinate System.

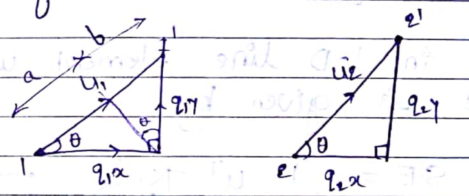
It is the Co-ordinate system in which the displacement of each node represents in two mutually ⊥r dirn. called as GCS.

Let q_1x, q_1y & q_2x, q_2y be the displ. of node 1 & 2 respt. in GCS. Displ. vector in GCS is

$$= \begin{bmatrix} q_1x \\ q_1y \\ q_2x \\ q_2y \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 2x \\ 2y \end{matrix}$$

(DOF; 4×1)

Hence representing displ. in LCS to be represented in the form of displ. of GCS



$$\cos\theta = \frac{a}{q_1x}$$

$$a = q_1x \cos\theta$$

$$\sin\theta = \frac{b}{q_1y}$$

$$b = q_1y \sin\theta$$

$$u_1 = a + b$$

$$= q_1x \cos\theta + q_1y \sin\theta \quad \text{—(1)}$$

$$u_2 = q_2x \cos\theta + q_2y \sin\theta \quad \text{—(2)}$$

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Matrix eqn ① ②

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}_{2\times4} \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \end{bmatrix}_{4\times1}$$

$$u_{2\times1} = C_{2\times4} \cdot q_{4\times1} \quad \text{——— ③}$$

where

$$C = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}_{2\times4}$$

S.E in 1-D line element using LCS is given by

$$S.E = \frac{1}{2} u^T_{1\times2} \cdot K_{2\times2} \cdot u_{2\times1} \quad \text{——— ④}$$

use $u = c.q$

$$= \frac{1}{2} (C.q)^T \cdot K \cdot C.q$$

$$= \frac{1}{2} q^T C^T \cdot K \cdot C.q$$

$$S.E = \frac{1}{2} q^T K^{(e)} q \quad \text{——— ⑤}$$

where

$$K^{(e)} = C^T \cdot K \cdot C$$

where eqn ⑤ represent the SE in global co-ordinate system & $K^{(e)}$ is element stiffness matrix for 1-D line element in GCS

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$$K^{(e)} = C^T \cdot K \cdot C$$

$$K^{(e)} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}_{4\times2} \times \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}$$

$$K^{(e)} = \frac{AE}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\cdot\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\cdot\sin\theta & \sin^2\theta & -\cos\theta\cdot\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\cdot\sin\theta & \sin^2\theta \end{bmatrix}_{4\times4}$$

use $\cos\theta = l$, $\sin\theta = m$.

$$K^{(e)} = \frac{AE}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}_{4\times4}$$

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Stress in truss element

$$\sigma^{(e)} = \frac{F^{(e)}}{A} = \frac{K.(u_2 - u_1)}{A}$$

$$= \frac{\frac{AE}{L}(u_2 - u_1)}{A}$$

$$= \frac{E}{L}(u_2 - u_1)$$

$$= \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix} \leftarrow \text{L.C.S}$$

$\because u = c \cdot q.$

$$= \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}_{1\times2} c_{2\times4} \cdot q_{4\times1}$$

$$= \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}_{1\times2}\begin{bmatrix}\cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta\end{bmatrix}_{2\times4} q_{4\times1}$$

$$= \frac{E}{L}\begin{bmatrix}-\cos\theta & -\sin\theta & \cos\theta & \sin\theta\end{bmatrix}_{1\times4} q_{4\times1}$$

$$\boxed{\sigma^{(e)} = \frac{E}{L}\begin{bmatrix}-l & -m & l & m\end{bmatrix}_{1\times4} \cdot q_{4\times1}}$$

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★ Pb. to be solved by co-ordinate approached.

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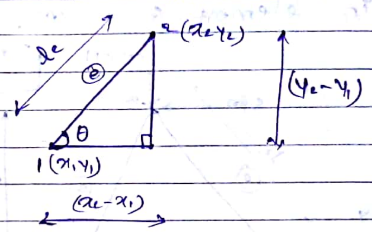
★ Co-ordinate approach.

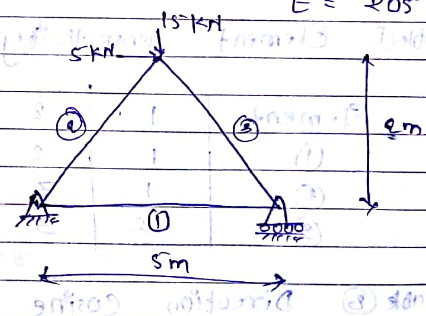


$$l = \cos\theta = \frac{(x_2 - x_1)}{l_e}$$

$$m = \sin\theta = \frac{(y_2 - y_1)}{l_e}$$

Q) Determine nodal displ., element stresses & support reaction for the 3-bar stress sh. in figure.

Area of each member $A = 1000\, mm^2$
$E = 205$ GPa



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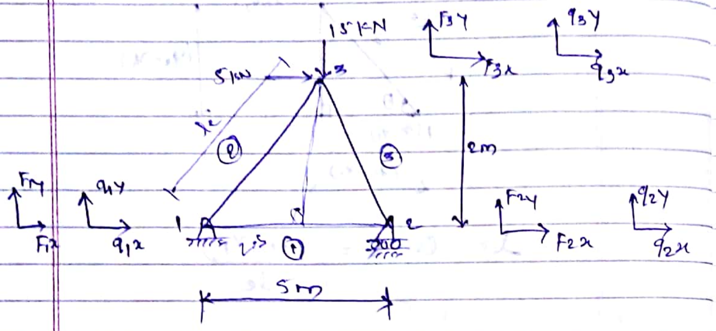
⇒ 1) Discritization the complete truss in element & nodes.



Where q_{1x} q_{1y} q_{2x} q_{2y} & q_{3x} q_{3y} be the displacement at node 1, 2, 3 respt. in x & y dirn.

F_{1x} F_{1y}, F_{2x} F_{2y} & F_{3x} F_{3y} be the nodal forces at node 1, 2, 3 respt in x & y dirn.

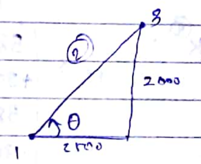
Table ① Element Conocctivity table.

Element	1	2	← Local node
①	1	2	Global node.
②	1	3	
③	2	3	

Table ② Direction cosine table. (4 a m) table.

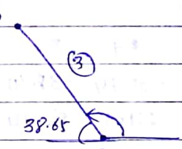
Element	le (mm)	l = cos θ	m sin θ	l^2	m^2	lm
1	5000	1	0	1	0	0
By P.T 2	3201.56 ~~5385~~	0.78	0.60	0.60	0.38	0.48
3	~~5385~~ 3201.56	-0.78	0.62	0.60	0.38	-0.48

θ = 0 or 360°



$\tan\theta = \frac{2000}{2500}$

θ = 38.65°



θ = 180 − 38.65
= 141.35°

2) Element stiffness matrix

$$K^{(e)} = \frac{AE}{l}\begin{bmatrix} L^2 & LM & -L^2 & -LM \\ LM & M^2 & -LM & -M^2 \\ -L^2 & -LM & L^2 & LM \\ -LM & -M^2 & LM & M^2 \end{bmatrix}$$

$$= \frac{1000 \times 205 \times 10^3}{5000} \begin{bmatrix} \\ \end{bmatrix}$$

$= 41 \times 10^3$

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$$K^{(1)} = 10^3 \begin{bmatrix} 41 & 0 & -41 & -0 \\ 0 & 0 & -0 & -0 \\ -41 & 0 & -41 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 2x \\ 2y \end{matrix} \; N/mm$$

(columns: 1x, 1y, 2x, 2y ← DOF)

640.3×10^3

$$K^{(2)} = 10^3 \begin{bmatrix} 38.41 & 30.74 & -38.41 & -30.74 \\ +30.74 & +24.3 & -30.74 & -24.3 \\ -38.41 & -30.74 & 38.41 & 30.74 \\ -30.74 & -24.3 & +30.74 & 24.3 \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 3x \\ 3y \end{matrix}$$

(columns: 1x, 1y, 3x, 3y)

$$K^{(3)} = 10^3 \begin{bmatrix} 38.41 & -30.74 & -38.41 & 30.74 \\ -30.74 & 24.3 & 30.74 & -24.3 \\ -38.41 & 30.74 & 38.41 & -30.74 \\ 30.74 & -24.3 & -30.74 & 24.3 \end{bmatrix} \begin{matrix} 2x \\ 2y \\ 3x \\ 3y \end{matrix}$$

(columns: 2x, 2y, 3x, 3y)

3) Global stiffness matrix

$$K = K^{(1)} + K^{(2)} + K^{(3)}$$

$$K = 10^3 \begin{bmatrix} 79.41 & 30.74 & -41 & 0 & -38.41 & -30.74 \\ 30.74 & 24.3 & -0 & -0 & -30.74 & -24.3 \\ -41 & 0 & 79.4 & -30.74 & -38.41 & 30.74 \\ 0 & 0 & -30.74 & 24.3 & 30.74 & -24.3 \\ -38.41 & -30.74 & -38.41 & 30.74 & 76.82 & 0 \\ -30.74 & -24.33 & 30.74 & -24.3 & 0 & 48.6 \end{bmatrix}_{6\times6} \begin{matrix} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{matrix}$$

(columns: 1x, 1y, 2x, 2y, 3x, 3y)

For check it is right add all rows or col. their sum should be zero & mirror image

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4) Nodal displacement.

As per PMPE

$$[F]_{6\times1} = [K]_{6\times6} \times [q]_{6\times1} \quad \text{———} \star$$

Using B.C.

$$10^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \\ -15 \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{matrix} = 10^3 \begin{bmatrix} 79.41 & 30.74 & -41 & 0 & 38.41 & -30.74 \\ 30.74 & 24.3 & 0 & 0 & -30.74 & -24.3 \\ -41 & 0 & 79.4 & -30.74 & -38.41 & 30.74 \\ 0 & 0 & 30.74 & -24.3 & 30.74 & -24.3 \\ -38.41 & -30.74 & -38.41 & 30.74 & 76.82 & 0 \\ -30.74 & -24.33 & 30.74 & -24.3 & 0 & 48.6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ q_{2x} \\ 0 \\ q_{3x} \\ q_{3y} \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 2x \\ 2y \\ 3x \\ 3y \end{matrix}$$

$79.4\, q_{2x} - 38.41\, q_{3x} + 30.74\, q_{3y} = 0$

$-38.41\, q_{2x} + 76.82\, q_{3x} + 0\, q_{3y} = 5$

$30.74\, q_{2x} + 0 + 48.6\, q_{3y} = -15$

$q_{2x} = 0.115 \quad 0.29$

$q_{3x} = -0.010 \quad 0.21$

$q_{3y} = -0.40 \quad -0.49$

Displacement matrix

$$q = \begin{bmatrix} 0 \\ 0 \\ 0.29 \\ 0 \\ 0.21 \\ -0.49 \end{bmatrix}$$

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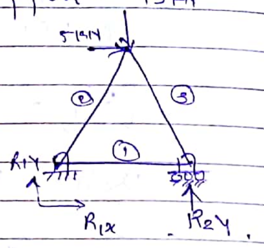
5] Reaction at support.



$R_{1x} =$ (1st row of k mat) $\times$ (q).

=

$R_{1y} =$ 5.45 KN (↑)

$R_{2y} =$ + 9.15 KN (↑)

6) Stresses in each element.

$\sigma^{①} = \frac{E}{l}[-l \ -m \ l \ m] \cdot q_{4\times1}$

$= \frac{205\times10^3}{5000}[-1 \ -0 \ 1 \ 0]\begin{bmatrix}0\\0\\0.29\\0\end{bmatrix}$

$= [-41 \ 0 \ 41 \ 0]\begin{bmatrix}0\\0\\0.29\\0\end{bmatrix}$

$\sigma^{①} = 11.89 \ N/mm^2$

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$\sigma^{②} = \frac{E}{L}(-l \ -m \ l \ m)\begin{bmatrix}q_{1x}\\q_{1y}\\q_{3x}\\q_{3y}\end{bmatrix}$

$= 64.03(-0.78 \ -0.60 \ 0.78 \ 0.60)$

$= [49.94 \ 38.41 \ -49.94 \ -38.41]\begin{bmatrix}0\\0\\0.29\\-0.49\end{bmatrix}$

$\sigma^{②} =$

$\sigma^{③} =$

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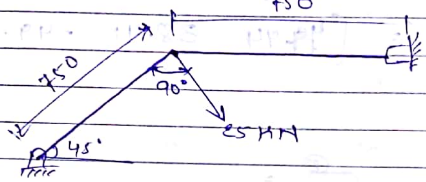
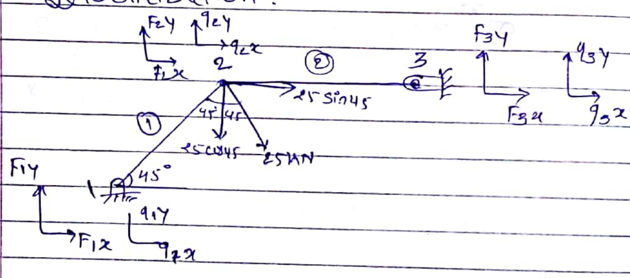
Q] Truss sh. in fig. with c/s area of all elements area = 500 mm² E = 2×10⁵ N/mm² determine 1) Nodal displ. 2) Stresses 3) Reaction force.



=) Discritization.



Element Connectivity table

Element	1	2
①	1	2
②	2	3

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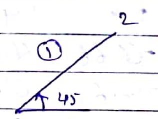
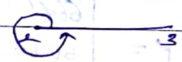
Direction Cosine table.

Element	L (mm)	l (cosθ)	m (sinθ)	l^2	m^2	lm
1	750	0.707	0.707	0.5	0.5	0.5
2	750	1	0	1	0	0





θ = 45°   θ = 0 or 360°

2) Element Stiffness

$$K^{(1)} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{500 \times 2\times10^5}{750}$$

$$= 10^3 \begin{bmatrix} 66.66 & 66.66 & -66.66 & -66.66 \\ 66.66 & 66.66 & -66.66 & -66.66 \\ -66.66 & -66.66 & 66.66 & 66.66 \\ -66.66 & -66.66 & 66.66 & 66.66 \end{bmatrix}$$

(columns: 1x 1y 2x 2y; rows: 1x 1y 2x 2y)

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$$K^{(2)} = 10^3 \begin{bmatrix} 133.33 & 0 & -133.33 & 0 \\ 0 & 0 & 0 & 0 \\ -133.33 & 0 & 133.33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(columns/rows: 2x, 2y, 3x, 3y)

③ Global stiffness matrix.

$K = 10^3$

	1x	1y	2x	2y	3x	3y	
	66.66	66.66	-66.66	-66.66	0	0	1x
	66.66	66.66	-66.66	-66.66	0	0	1y
	-66.66	-66.66	199.99	66.66	-133.33	0	2x
	-66.66	-66.66	66.66	66.66	0	0	2y
	0	0	-133.33	0	133.33	0	3x
	0	0	0	0	0	0	3y

4) Nodal displ.

$[F] = (K)(q)$

$$\begin{bmatrix} 0 \\ 0 \\ 17.67 \\ -17.67 \\ 0 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} 66.66 & 66.66 & -66.66 & -66.66 & 0 & 0 \\ 66.66 & 66.66 & -66.66 & -66.66 & 0 & 0 \\ -66.66 & -66.66 & 199.99 & 66.66 & -133.33 & 0 \\ -66.66 & -66.66 & 66.66 & 66.66 & 0 & 0 \\ 0 & 0 & -133.33 & 0 & 133.33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ q_{2x} \\ q_{2y} \\ 0 \\ 0 \end{bmatrix}$$

$199.99\, q_{2x} + 66.66\, q_{2y} = 17.67$

$66.66\, q_{2x} + 66.66\, q_{2y} = -17.67$

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$q_{2x} = 0.26$

$q_{2y} = -0.53$

4. Disp. matr

$$\begin{bmatrix} 0 \\ 0 \\ 0.26 \\ -0.53 \\ 0 \\ 0 \end{bmatrix}$$

⑤. Reaction @ support

$R_{1x} = 17.99$ kN

$R_{1y} =$

$R_{3x} = -34.66$ (kN) (←)

$R_{3y} = 0$ (kN)

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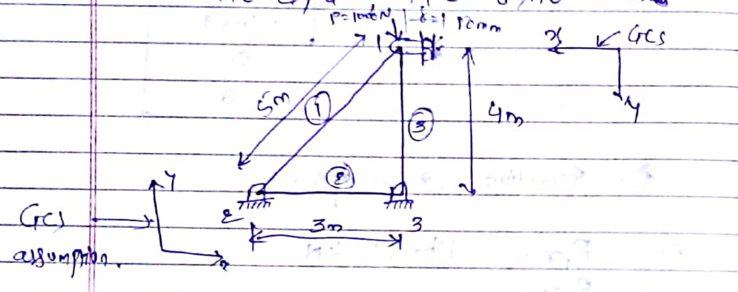
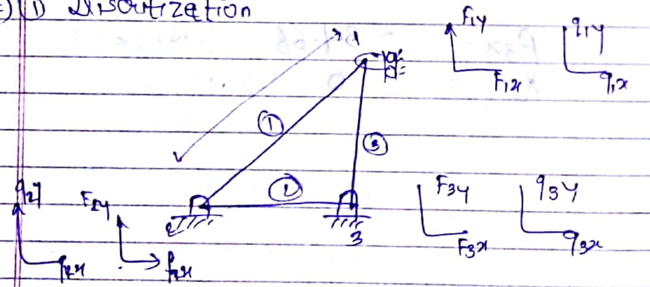
Q. For the truss sh. in fig. determine displ. in Y dirn of node '1' & axial force of P = 1000 kN is applied at node 1 in the Y dirn, while node 1 settle an amt δ = 50 mm in the -ve x-dirn.

E = 210 GPa A = 6×10^{-4} m²



1) Discritization



Element connectivity table.

Element	1	2
1	1	2
2	2	3
3	1	3

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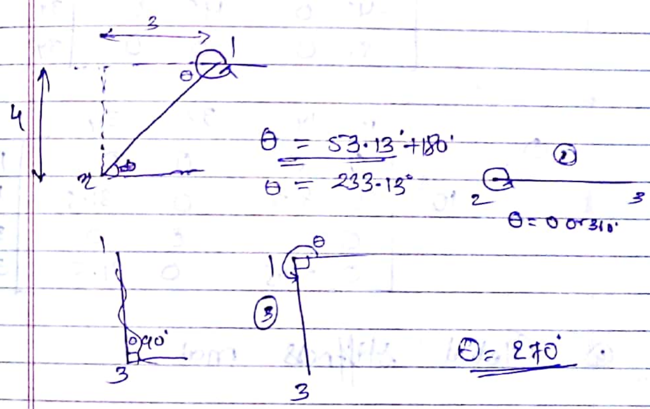
Dirn cosine table.

Element	L (mm)	l (cos θ)	m (sin θ)	l^2	m^2	lm
1	5000	-0.60	-0.879	0.36	0.62	0.487
2	3000	1	0	1	0	0
3	4000	0	-1	0	1	0



2) Element stiffness.

$$K = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{600 \times 210 \times 10^3}{5000}\begin{bmatrix} & \end{bmatrix}$$

Long Book $= 25.2 \times 10^3$

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	1x	1y	2x	2y	
$= 10^3$	9.072	11.84	-9.072	-11.84	1x
	11.84	15.62	-11.84	-15.62	1y
	-9.072	-11.84	9.072	11.84	2x
	-11.84	-15.62	11.84	15.62	2y

$K^{(2)} = 10^3$

2x	2y	3x	3y	
42	0	-42	0	2x
0	0	0	0	2y
-42	0	42	0	3x
0	0	0	0	3y

$K^{(3)} = 10^3$

1x	1y	3x	3y	
0	0	0	0	1x
0	31.5	0	-31.5	1y
0	0	0	0	3x
0	-31.5	0	31.5	3y

③ Global stiffness mat.

K =

1x	1y	2x	2y	3x	3y	
9.072	11.84	-9.072	-11.84	0	0	1x
11.84	47.12	-11.84	-15.62	0	-31.5	1y
-9.072	-11.84	51.072	11.84	-42	0	2x
-11.84	-15.62	11.84	15.62	0	0	2y
0	0	-42	0	42	0	3x
0	-31.5	0	0	0	31.5	3y

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4) Nodal displ.

$[F] = [K][q]$

$$\begin{bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} \; \end{bmatrix} \times \begin{bmatrix} 0 \\ q_{1y} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$q_{1y} = \frac{-1000}{47.12} = -21.27 \text{ mm}$$

Displacement in y- dirⁿ of node 1 is -21.27 mm

$$F^{(1)} = \frac{AE}{L} [-l \;\; -m \;\; l \;\; m]_{1\times4} [q]_{4\times1}$$

$$= \frac{600 \times 210 \times 10^3}{5000} [+0.60 \;\; +0.79 \;\; -0.60 \;\; -0.79] \begin{bmatrix} 0 \\ q_{1y} \\ 0 \\ 0 \end{bmatrix}$$

$$= 25.2 [0.60 \;\; 0.79 \;\; -0.60 \;\; -0.79] \begin{bmatrix} 0 \\ -21.27 \\ 0 \\ 0 \end{bmatrix}$$

$A = 200\ mm^2$ $E = 200 \times 10^3\ mpa$

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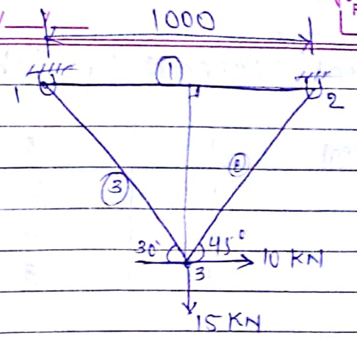
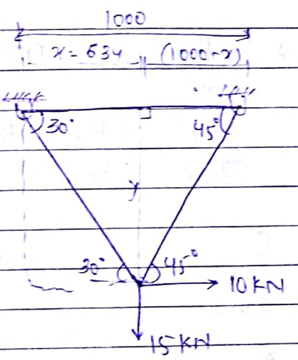
Q14]



1) Discritization



$L_2 = \sqrt{(1000-x)^2 + y^2}$

$L_3 = \sqrt{y^2 + x^2}$

$\tan 30 = \frac{y}{x}$

$y = x \tan 30$ —①

$\tan 45 = \frac{y}{(1000-x)}$

$y = (1000-x) \tan 45$ —②

① = ② ⇒

$x = 634\ mm$

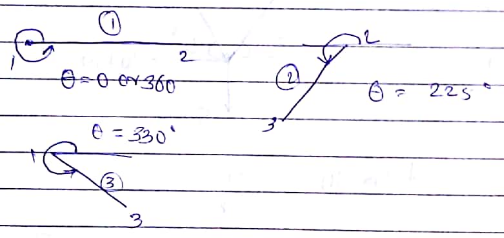
$y = 366\ mm$

Table ① Element connectivity table.

Element	1	2
①	1	2
②	2	3
③	1	3

Table ② Dir^n cosine table (l&m) table.

Element	L	l	m	l^2	m^2	lm
1	1000	1	0	1	0	0
2	517.63	-0.707	-0.707	0.5	0.5	0.5
3	732.06	0.86	-0.5	0.75	0.25	-0.43



Or

If asked solve by nodal co-ordinate ~~tabl.~~ approach then solve this ⟶

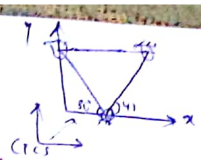


Table ① Nodal co-ordinate table.

Nodes	x	y
1	0 x_1	366 y_1
2	1000 x_2	366 y_2
3	634	0

Table ② Element connectivity table.

Element	1	2
①	1	2
②	2	3
③	1	3

Table ③ Dir^n cosine table.

Element	L	$l = \left(\frac{x_2 - x_1}{L}\right)$	$m = \left(\frac{y_2 - y_1}{L}\right)$	l^2	m^2	lm
①	1000	1	0	1	0	0
②	517.63	-0.707	-0.707	0.5	0.5	0.5
③	732.06	0.86	-0.5	0.75	0.25	-0.43

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4) Element stiffness table.

$$K^{(1)} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$= 10^3$

1x	1y	2x	2y	
40	0	-40	0	1x
0	0	0	0	1y
-40	0	40	0	2x
0	0	0	0	2y

$K^{(2)} = 10^3$

2x	2y	3x	3y	
38.63	38.63	-38.63	-38.63	2x
38.63	38.63	-38.63	-38.63	2y
-38.63	-38.63	38.63	38.63	3x
-38.63	-38.63	38.63	38.63	3y

$K^{(3)} = 10^3$

1x	1y	3x	3y	
40.98	-23.49	-48.98	23.49	1x
-23.49	13.66	23.49	-13.66	1y
-48.98	23.49	40.98	-23.49	3x
23.49	-13.66	-23.49	13.66	3y

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5) Global stiffness table.

1x	1y	2x	2y	3x	3y	
80.98	-23.49	-40	0	-48.98	23.49	1x
-23.49	13.66	0	0	23.49	-13.66	1y
-40						2x
						2y
						3x
						3y

If more than 3 member than by Symmetric method

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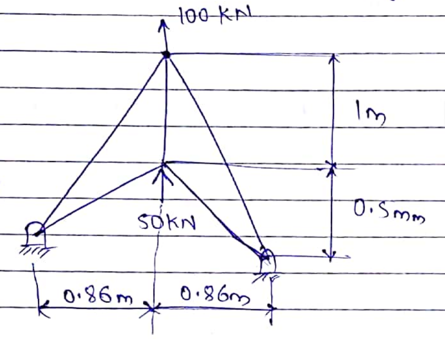
Symmetric truss

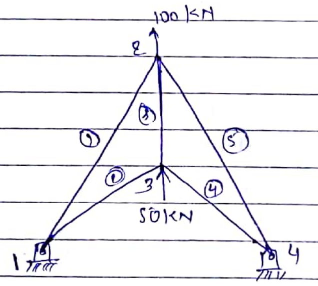
Q1) Determine displacement field, stresses & strain & reaction at support for the 2-D truss sh. in fig.



$A = 200 mm^2$

$E = 2 \times 10^5$ mpa

⇒ 1) Discritization.



As the given structure contoert is exactly symmetric about vertical axis hence considering only half of the part. of the truss for FEM as sh. in fig

In symmetric only vertical load are to be carried out

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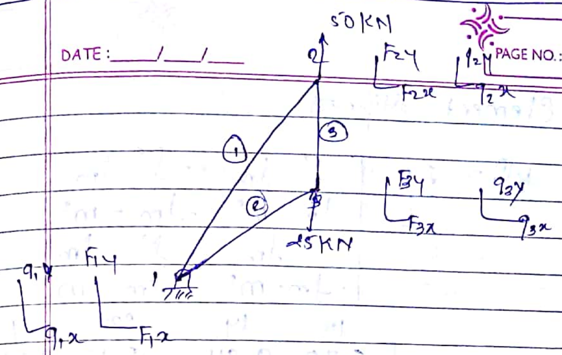
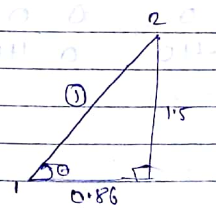


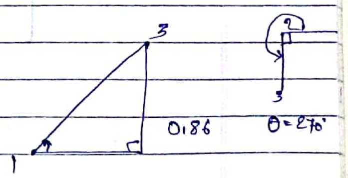
Table 1] Element connectivity table.

Elements	1	2
①	1	2
②	1	3
③	2	3

Table 2) Dir^n Cosine table.

Element	L	$l = \cos\theta$	$m = \sin\theta$	l^2	m^2	lm
①	1729.05	0.5	0.86	0.25	0.75	0.43
②	994.79	0.86	0.5	0.75	0.25	0.43
③	1000	0	-1	0	1	0





$\theta = \tan^{-1}\left(\frac{1.5}{0.86}\right)$

$= 60.17°$

$\theta = \tan^{-1}\left(\frac{0.5}{0.86}\right)$

$= 30.17°$

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③ Element Stiffness.

$$K^{①} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 5.78 & 9.94 & -5.78 & -9.94 \\ 9.94 & 17.34 & -9.94 & -17.34 \\ -5.78 & -9.94 & 5.78 & 9.94 \\ -9.94 & -17.34 & 9.94 & 17.34 \end{bmatrix}$$

(columns: 1x, 1y, 2x, 2y; rows: 1x, 1y, 2x, 2y)

$$K^{②} = 10^3 \begin{bmatrix} 30.15 & 17.28 & -30.15 & -17.28 \\ 17.28 & 10.05 & -17.28 & -10.05 \\ -30.15 & -17.28 & 30.15 & 17.28 \\ -17.28 & -10.05 & 17.28 & 10.05 \end{bmatrix}$$

(columns: 1x, 1y, 3x, 3y; rows: 1x, 1y, 3x, 3y)

$$K^{③} = 10^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 40 & 0 & -40 \\ 0 & 0 & 0 & 0 \\ 0 & -40 & 0 & 40 \end{bmatrix}$$

(columns: 2x, 2y, 3x, 3y; rows: 2x, 2y, 3x, 3y)

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④ Global stiffness matrix.

$$= 10^3 \begin{bmatrix} 35.93 & 27.22 & -5.78 & -9.94 & -30.15 & -17.28 \\ 27.22 & 27.39 & -9.94 & -17.34 & -17.28 & -10.05 \\ -5.78 & -9.94 & 5.78 & 9.94 & 0 & 0 \\ -9.94 & -17.34 & 9.94 & 57.34 & 0 & -40 \\ -30.15 & -17.28 & 0 & 0 & 30.15 & 10.05 \\ -17.28 & -10.05 & 0 & -40 & 10.05 & 50.05 \end{bmatrix}$$

(columns: 1x, 1y, 2x, 2y, 3x, 3y; rows: 1x, 1y, 2x, 2y, 3x, 3y)

⑤ Nodal displacement matrix

$$[F] = [K][q]$$

$$10^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \\ 0 \\ 25 \end{bmatrix} = 10^3 \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ q_{2y} \\ 0 \\ q_{3y} \end{bmatrix}$$

(columns: 1x, 1y, 2x, 2y, 3x, 3y; rows: 1x, 1y, 2x, 2y, 3x, 3y)

$$57.34\, q_{2y} - 40\, q_{3y} = 50$$
$$-40\, q_{2y} + 50.05\, q_{3y} = 25$$

$$q_{2y} = 2.75 \text{ mm}$$

$$q_{3y} = 2.70 \text{ mm}$$

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6) Stresses

$$\sigma^{(1)} = \frac{E}{l}\begin{bmatrix} -l & -m & l & m \end{bmatrix} \cdot q$$

$$= \frac{2\times10^5}{1729.05}\begin{bmatrix} -0.5 & -0.86 & 0.5 & 0.86 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.75 \end{bmatrix}$$

$$= 115.67 \times 2.365$$

$$\sigma^{(1)} = 273.55 \ N/mm^2$$

$$\sigma^{(2)} = 201.04 \times 1.35 = 271.40 \ N/mm^2$$

$$\sigma^{(3)} = \text{~~200 × 7.425~~} - \quad -10 \ N/mm^2$$

$$= \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 2.75 \\ 0 \\ 2.70 \end{bmatrix}$$

$$\frac{2\times10^5}{1000}(2.75-2.7)$$

$$\frac{2\times10^5}{1000}(0.05)$$

7) Reaction at support.

$R_{1x} = -74$ KN

$R_{1y} = -74.82$ KN

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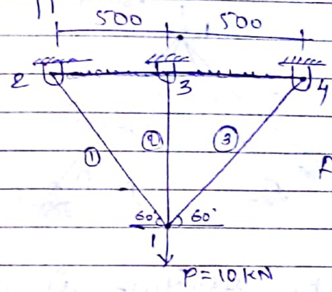
(Q)

Stresses in each member & reaction at support

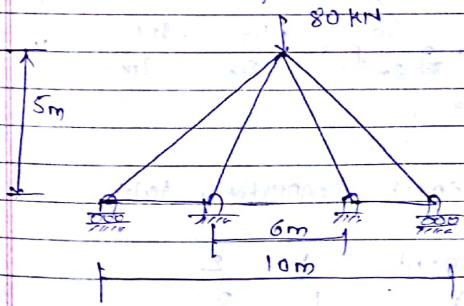


$A_1 = A_3 = 25 \ mm^2$

$A_2 = 50 \ mm^2$

$E = 70$ Gpa.

Q) ⇒ Determine nodal displ. matrices, stresses in each member for the truss sh. Also comment about the symmetry of the structure



$A = 1200 \ mm^2$

$E = 200$ Gpa

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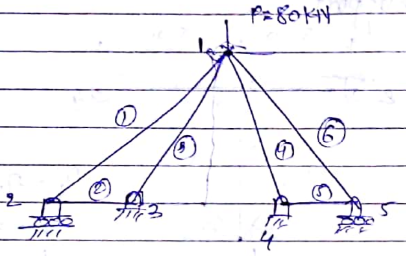
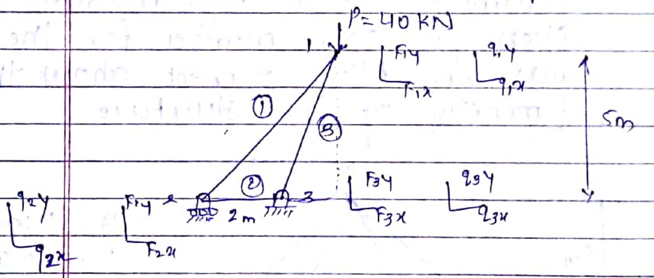
⇒ 1) Discritization

Discritizing the given structure into 6 1-D line elem. as sh. in fig.



As the structure is about symmetric axis hence considering half of truss



② Element connectivity table

Element	1	2
①	1	2
②	2	3
③	1	3

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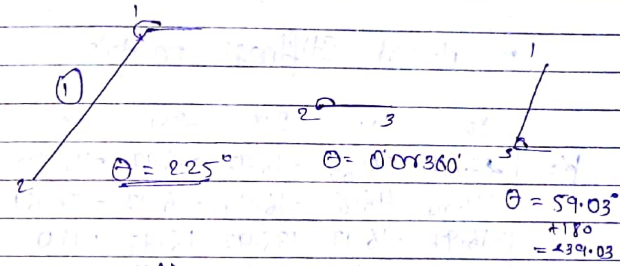
Dirn cosine table.

Element	L	l	m	l^2	m^2	lm
①	7071	-0.707	-0.707	0.5	0.5	0.5
②	2000	1	0	1	0	0
③	5830	-0.51	-0.857	0.26	0.72	0.43



• Element stiffness matrix

$$K^{①} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ l^2 & lm & l^2 & lm \\ lm & m^2 & lm & m^2 \end{bmatrix}$$

(columns: 1x, 1y, 2x, 2y; rows: 1x, 1y, 2x, 2y)

$$= 10^3 \begin{bmatrix} 16.97 & 16.97 & -16.97 & -16.97 \\ 16.97 & 16.97 & -16.97 & -16.97 \\ -16.97 & -16.97 & 16.97 & 16.97 \\ -16.97 & -16.97 & 16.97 & 16.97 \end{bmatrix}$$

(columns: 2x, 2y, 3x, 3y; rows: 2x, 2y, 3x, 3y)

$$K^{②} = 10^3 \begin{bmatrix} 120 & 0 & -120 & 0 \\ 0 & 0 & 0 & 0 \\ -120 & 0 & 120 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K^{(3)} = 10^3 \begin{bmatrix} 10.70 & 17.69 & -10.70 & -17.69 \\ 17.69 & 29.63 & -17.69 & -29.63 \\ -10.70 & -17.69 & 10.70 & 17.69 \\ -17.69 & -29.63 & 17.69 & 29.63 \end{bmatrix}$$

(columns/rows: 1x, 1y, 3x, 3y)

- Global Stiffness matrix.

	1x	1y	2x	2y	3x	3y	
K=	27.67	34.66	-16.97	-16.97	-10.70	-17.69	1x
	34.66	46.6	-16.97	-16.97	-17.69	-29.63	1y
	-16.97	-16.97	136.97	16.97	-120	0	2x
	-16.97	-16.97	16.97	16.97	0	0	2y
	-17.69	-17.69	-120	0	130.70	17.69	3x
	-17.69	-29.63	0	0	17.69	29.63	3y

- Nodal displacement matrix.

Load		1x	1y	2x	2y	3x	3y	q
0								0
-40	=		46.6	-16.97				q_1y
0			-16.97	136.97				q_2x
0								0
0								0
0								0

$$46.6\, q_1y - 16.97\, q_2x = -40$$
$$-16.97\, q_1y + 136.97\, q_2x = 0$$

$$q_1y = -0.99$$
$$q_2x = -0.11$$

Reaction of support

R_{2y} = ~~16.97~~ 13.81 KN

R_{3x} =

R_{3y} =

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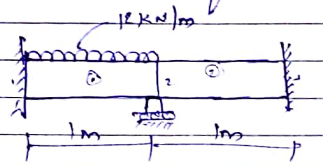
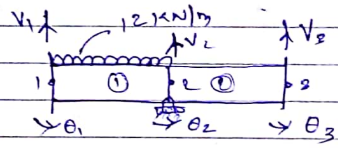
2(a) A beam sh. in fig is fixed at both the ends & supported betn the ends with a simple support that allows rotation, find
1) The rotation at a simple support
2) Reactions at all the supports.
3) Shear force & B.M. Also construct the diag.
4) Also determine the displ. at 0.5m from left support.



$E = 200 \times 10^9 \ N/m^2$

$I = 4 \times 10^{-6} m^4$

⇒ 1) Discritization.



2) Element stiffness matrix.

$$K^{①} = 0.8 \begin{bmatrix} 9.6 & 4.8 & -9.6 & 4.8 \\ 4.8 & 3.2 & -4.8 & 1.6 \\ -9.6 & -4.8 & 9.6 & -4.8 \\ 4.8 & 1.6 & -4.8 & 3.2 \end{bmatrix} \begin{matrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{matrix}$$

(columns: V_1, θ_1, V_2, θ_2)

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⟶ Contd →

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Beam element

The element stiffness matrix for beam element is given by

$$K^{e} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{matrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{matrix}$$

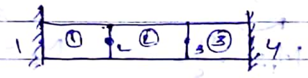
(columns: V_1, θ_1, V_2, θ_2)

Where,
e = modulus of elasticity (N/m² or kN/m²)
I = MOI (m^4)
L = Length of element.

Finite element formulation

Deflection at any location i.e nodal displ. & slope can also be determined by using FEM.

Consider a beam



Here each node is having 2 DOF i.e 1 displacement is in transverse dirn & other slope or rotation.
In beam element upward transverse dirn displacement is considered as +ve & slope or

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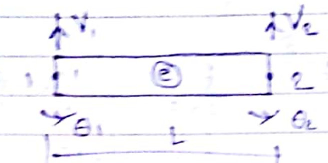
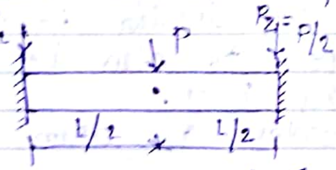
Rotation in CCW is considered as +ve. Consider a single element of a beam as sh in fig.



∵ the displ. vector, $q = \begin{bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{bmatrix}_{4\times1}$

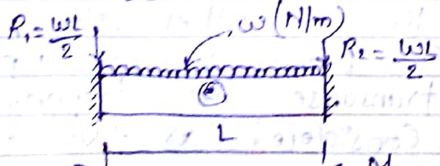
★ <u>Equivalent joint loading condn.</u>

Case 1] A beam is s.t a pt. load.



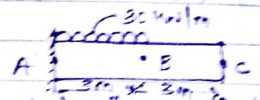
$M_1 = \frac{PL}{8}$ $M_2 = \frac{PL}{8}$

Case 2] If the beam is s.t to UDL



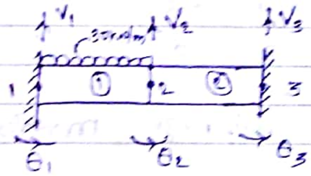
$M_1 = WL^2/12$ $M_2 = \frac{WL^2}{12}$

Q1] Det. nodal displ. & reaction for the beam as sh. in fig. Take $E \times I = 9000$ kN-m



Also determine displ. at the mid pt. of an UDL for section AB.

⇒ 1) Discritization



Where
$V_1 V_2 V_3 \rightarrow$ transverse defln at node 123
$\theta_1 \theta_2 \theta_3 \rightarrow$ slope or rotation at node 123 respt.

2) Element stiffness matrix

$$K^{①} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{matrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{matrix}$$

(columns: V_1 θ_1 V_2 θ_2)

$= 333.33 \begin{bmatrix} - \end{bmatrix}$

$= \frac{333.33}{1000}$

$= 0.333 \begin{bmatrix} \end{bmatrix}$

$$K^{(1)} = \begin{bmatrix} 4 & 6 & -4 & 6 \\ 6 & 12 & -6 & 6 \\ -4 & -6 & 4 & 6 \\ 6 & 6 & -6 & 12 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

(columns: v_1, θ_1, v_2, θ_2)

$$K^{(2)} = \begin{bmatrix} 4 & 6 & -4 & 6 \\ 6 & 12 & -6 & 6 \\ -4 & -6 & 4 & 6 \\ 6 & 6 & -6 & 12 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(columns: v_2, θ_2, v_3, θ_3)

3) Global stiffness matrix

$$K = \begin{bmatrix} 4 & 6 & -4 & 6 & 0 & 0 \\ 6 & 12 & -6 & 6 & 0 & 0 \\ -4 & -6 & 8 & 12 & -4 & 6 \\ 6 & 6 & 0 & 24 & -6 & 6 \\ 0 & 0 & -4 & -6 & 4 & 6 \\ 0 & 0 & 6 & 6 & -6 & 12 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

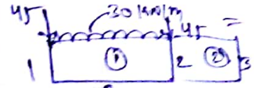
(columns: v_1, θ_1, v_2, θ_2, v_3, θ_3)

4) Nodal displ. matrix

As per PMPE's

$[F] = [K][q]$

Using B.C



$= 30 \times 3 = 90$ kN/m

$$\begin{bmatrix} -45 \\ -22.5 \\ -45 \\ 22.5 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & 8 & 12 & & \\ & & 0 & 24 & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(left labels: R_1, M_1, R_2, M_2; columns: v_1, θ_1, v_2, θ_2, v_3, θ_3)

$8 v_2 + 12 \theta_2 = -45$

$0 v_2 + 24 \theta_2 = 22.5$

$v_2 = -7.031$ m

$\theta_2 = 0.9375$ rad

5) Reaction

$$R^{(1)} = \begin{bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} = K \cdot q - F$$

(DOF)

$$= 10^{3} \begin{bmatrix} 4 & 6 & -4 & 6 \\ 6 & 12 & -6 & 6 \\ -4 & -6 & 8 & 12 \\ 6 & 6 & 0 & 24 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ -7.031 \\ 0.9375 \end{bmatrix} - \begin{bmatrix} -45 \\ -22.5 \\ -45 \\ 22.5 \end{bmatrix}$$

(columns: v_1, θ_1, v_2, θ_2)

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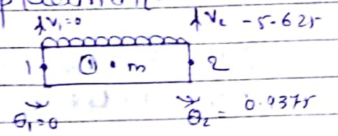
$$R^{①} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \quad R^{②} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Now displacement



$$V(\xi_1) = H_1 v_1 + \frac{l_e}{2} H_2 \theta_1 + H_3 v_2 + \frac{l_e}{2} H_4 \theta_2$$

$$V(\xi_1) = -3.16 \times 10^{-3} \text{ m}$$

2nd pt. contd.

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$$K^{②} = \begin{bmatrix} 9.6 & 4.8 & -9.6 & 4.8 \\ 4.8 & 3.2 & -4.8 & 1.6 \\ -9.6 & -4.8 & 9.6 & -4.8 \\ 4.8 & 1.6 & -4.8 & 3.2 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(columns: v_2, θ_2, v_3, θ_3)

③ Global stiffness

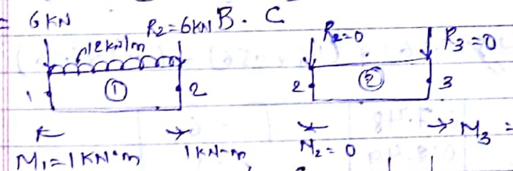
$$K = \begin{bmatrix} 9.6 & 4.8 & -9.6 & 4.8 & 0 & 0 \\ 4.8 & 3.2 & -4.8 & 1.6 & 0 & 0 \\ -9.6 & -4.8 & 19.2 & 0 & -9.6 & 4.8 \\ 4.8 & 1.6 & 0 & 6.4 & -4.8 & 1.6 \\ 0 & 0 & -9.6 & -4.8 & 9.6 & -4.8 \\ 0 & 0 & 4.8 & 1.6 & -4.8 & 3.2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(columns: v_1, θ_1, v_2, θ_2, v_3, θ_3)

4) Nodal displ.

$$[F] = [K] \cdot [q]$$

B.C



$$\begin{bmatrix} -6 \\ -1 \\ -6 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & 6.4 & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

(columns: v_1, θ_1, v_2, θ_2, v_3, θ_3)

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$\theta_2 = 1.56 \times 10^{-4}$ rad

$$q = 10^{-4}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.56 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \text{ rad} \\ V_3 \\ \theta_3 \end{matrix}$$

3) Reactions

$R_1 = K_{(1)} \cdot q - F$

$$= \begin{bmatrix} V_1 & \theta_1 & V_2 & \theta_2 \\ 9.6 & 4.8 & -9.6 & 4.8 \\ 4.8 & 3.2 & -4.8 & 1.6 \\ -9.6 & -4.8 & 19.2 & 0 \\ 4.8 & 1.6 & 0 & 6.4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.56 \end{bmatrix} - \begin{bmatrix} -6 \\ -1 \\ -6 \\ 1 \end{bmatrix}$$

$(4.8 \times 1.56) + (1.6 \times 1.56) + (6.4 \times 1.56)$

$$\begin{bmatrix} 0.7.48 \\ 0.249 \\ 0 \\ 9.98 \end{bmatrix} - \begin{bmatrix} -6 \\ -1 \\ -6 \\ 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 13.48 \\ 3.49 \\ 6 \\ 8.98 \end{bmatrix}$$

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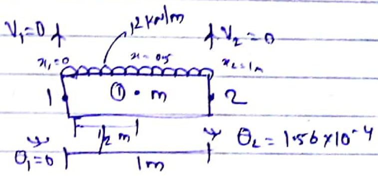
$R_2 = K_{(2)} \cdot q - F$

$$= \begin{bmatrix} 9.6 & 4.8 & -9.6 & 4.8 \\ 4.8 & 3.2 & -4.8 & 1.6 \\ -9.6 & -4.8 & 9.6 & -4.8 \\ -4.8 & 1.6 & -4.8 & 3.2 \end{bmatrix} 10^3 \times \begin{bmatrix} 0 \\ 1.56 \\ 0 \\ 0 \end{bmatrix} \times 10^{-4}$$

$(4.8 \times 10^3 \times 1.56 \times 10^{-4}) + (3.2 \times 10^3 \times 1.56 \times 10^{-4}) - (4.8 \times 10^3 \times 1.56 \times 10^{-4}) + 1.6 \times 10^3 \times 1.56 \times 10^{-4}$

$$R_2 = \begin{bmatrix} 4.64 \\ 3.004 \\ 4.64 \end{bmatrix} = \begin{bmatrix} 0.748 \\ 0.5 \\ -0.74 \\ 0.256 \end{bmatrix}$$

Ans 4)



The displ. of element 1 from 0.5 m from left end is given by

$V(\xi_p) = H_1 V_1 + \frac{le}{2} H_2 \theta_1 + H_3 V_2 + \frac{le}{2} H_4 \theta_2$

Where

$H_1 = \frac{1}{4}(1-\xi_p)^2 (2+\xi_p)$

$= 0.5$

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$$H_2 = \frac{1}{4}(1-\xi_1)^2(\xi_1+1)$$

$$= 0.25$$

$$H_3 = 1/4\,(1+\xi_1)^2(2-\xi_1) = 0.5$$

$$H_4 = 1/4\,(1+\xi_1)^2(\xi_1-1) = -0.25$$

$$\boxed{V(\xi_1) = -1.95 \times 10^{-5}\ m}$$

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Imp

Hermite Shape function.

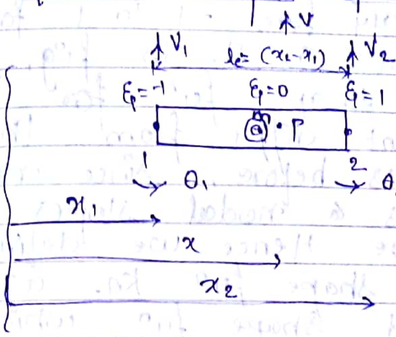

fig:→ Beam element.

The beam element is having two nodes, each node has 2 DOF.

The displacement vector for an element is

$$q = [V_1\ \theta_1\ V_2\ \theta_2]^T \quad \text{--- (1)}$$

The shape fun for interpolationg v i.e transverse displ. at any pt. within an element are defined in terms of natural co-ordinate (ξ_1) & is given by

$$x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \cdot \xi_1 \quad \text{--- (2)}$$

use $x = x_1$, $\xi_1 = -1$

use $x = x_2$, $\xi_1 = 1$

use $x = (x_1 + x)$ (mid pt.), $\xi_1 = 0$

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from the result it is clear that ξ vary from -1 to 1 for beam element as sh in fig.

The shape fun for beam element differ from those disscuss before, since nodal values & nodal slopes are involve. Hence we define a new shape fun kn. as hermit shape fun which satisfy nodal values & slope continuity requirements & is given by

$$H_1 = \frac{1}{4}(1-\xi)^2(2+\xi)$$

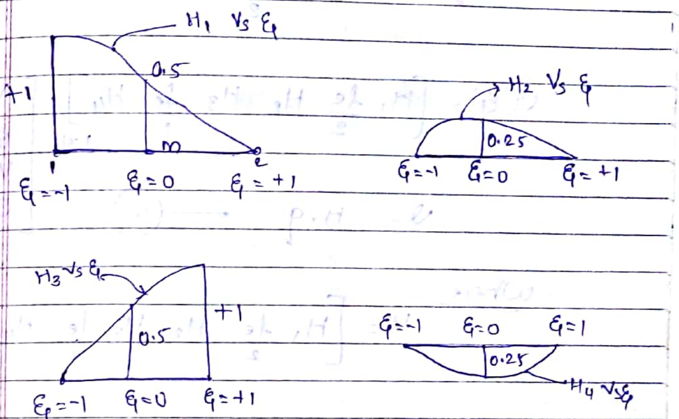
$$H_2 = \frac{1}{4}(1-\xi)^2(\xi+1)$$

$$H_3 = \frac{1}{4}(1+\xi)^2(2-\xi) \quad \text{—— (3)}$$

$$H_4 = \frac{1}{4}(1+\xi)^2(\xi-1)$$

Where H_1 H_2 H_3 H_4 kn. as hermit shape fun. out of which H_1 & H_3 represents for nodal values, whereas H_4 & H_2 represent for slope.

Plot the graph betn shape fun & natural coordinates.



from the above graph it is clear that the addn of shape fun at any pt. is equal to 1

i.e $H_1 + H_2 + H_3 + H_4 = 1$ —— (4)

Once the shape fun are defined the displacement at any pt. within an element is given by

$$v = H \cdot q$$

~~$v(\xi) = H_1 \frac{l_e}{2}$~~

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$$V = H \cdot q$$

$$v(\xi_1) = H_1 v_1 + \frac{l_e}{2} H_2 \theta_1 + H_3 v_2 + \frac{l_e}{2} H_4 \theta_2$$

$$v(\xi_1) = \begin{bmatrix} H_1 & \frac{l_e}{2} H_2 & H_3 & \frac{l_e}{2} H_4 \end{bmatrix}_{1\times4} \cdot \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}_{4\times1}$$

$$v = H \cdot q \quad \text{——— (5)}$$

Where

$$H = \begin{bmatrix} H_1 & \frac{l_e}{2} H_2 & H_3 & \frac{l_e}{2} H_4 \end{bmatrix}$$

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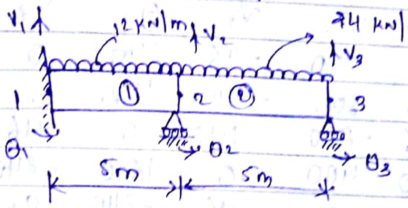
$E = 200$ GPa $= \frac{200 \times 10^9}{10^3} = 200 \times 10^6$ kN/m²

$I = 5 \times 10^6$ mm⁴ $= \frac{5 \times 10^6}{(10^3)^4} = 5 \times 10^{-12} \times 10^6 = 5 \times 10^{-6}$ m⁴

⇒ 1) Discritization.

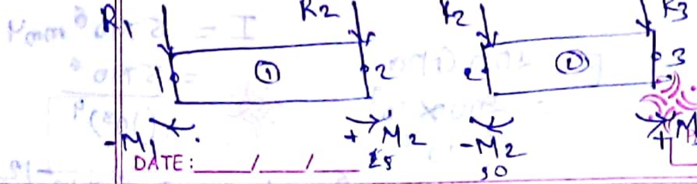


2) Element Stiffness matrix

$$K^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$K^{(1)} = \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 8000 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$K^{(2)} = \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 96 & 240 & -96 & 240 \\ 240 & 800 & -240 & 400 \\ -96 & -240 & 96 & -240 \\ 240 & 400 & -240 & 800 \end{bmatrix} \end{array} \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$



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3) Global stiffness mat.

	V_1	θ_1	V_2	θ_2	V_3	θ_3	
K =	96	240	-96	240	0	0	V_1
	240	800	-240	400	0	0	θ_1
	-96	-240	192	0	-96	240	V_2
	240	400	0	1600	-240	400	θ_2
	0	0	-96	-240	96	-240	V_3
	0	0	240	400	-240	800	θ_3

4) Nodal displ.

$[F] = [K][q]$

F		=	V_1	θ_1	V_2	θ_2	V_3	θ_3	q	
~~-30~~	V_1								0	V_1
~~-25~~	θ_1								0	θ_1
-90	V_2	=			192		-96		q_2x	V_2
~~-25~~	θ_2								0	θ_2
-60	V_3				-96		96		q_3x	V_3
~~-50~~	θ_3								0	θ_3

$192\, q_2x - 96\, q_3x = -90$

$-96\, q_2x + 96\, q_3x = -60$

$q_2x =$ m

$q_3x =$ m

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q: $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

5) Reactions

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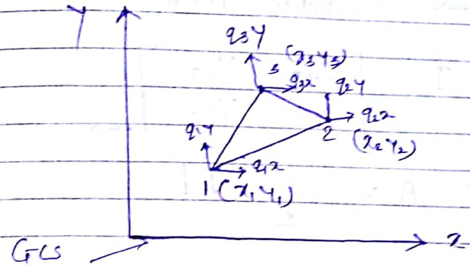
Unit 3

CST Element & Multipoint Constraint of 1-D element



CST → Const strain Triangle / 2-D triangular element.

Element stiffn. mat. for CST element or 2-D triangular element is given by

$$K^{e}_{6\times6} = t \cdot A \cdot B^{T}_{6\times3} \cdot D_{3\times3} \cdot B_{3\times6}$$

Where

- t → thickness of the element
- A → Surface area of CST elem.
- B → Strain displ relation mat
- D → Strain transformation mat.

Where

$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

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$J \rightarrow$ Jacobian matrix

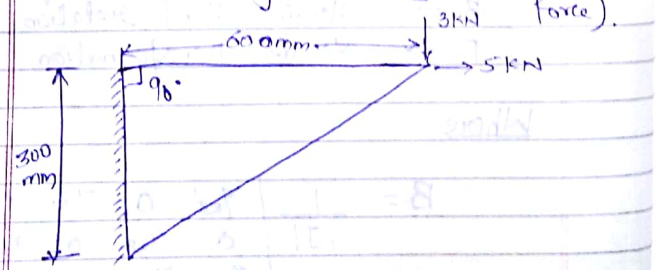
- $J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}_{2\times 2}$

- $\therefore A = \frac{1}{2} |J|$

- $D = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix}$

$E \rightarrow$ modulus of elasticity.
$\mu \rightarrow$ poisions ratio.

Q] A 2-D plate sh. in fig. The thickness of plate is 10mm & $E = 2\times10^5$ N/mm² & $\mu = 0.3$. Determine the nodal displ. using plane stress condⁿ. (neglect body & friction force).



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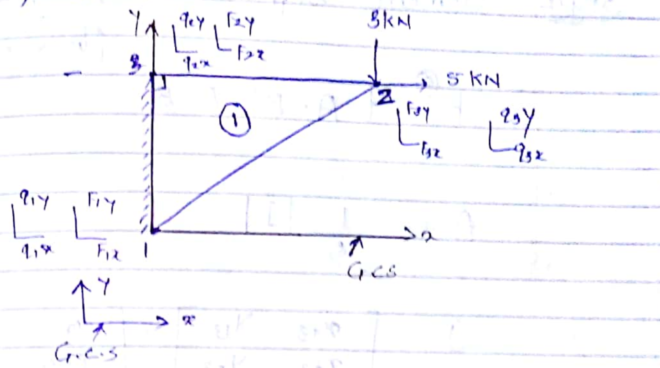
⇒ 1) Discritization



Let $F_{1x}, F_{1y}, F_{2x}, F_{2y}$ & F_{3x}, F_{3y} be the nodal forces in x & y dirⁿ respt.
Let $q_{1x}, q_{1y}, q_{2x}, q_{2y}, q_{3x}, q_{3y}$ be the nodal displ. in x & y dirⁿ respt. of nodes 1, 2 & 3 respt.

Table ① Nodal co-ordinate table

Nodes	x	y
1	x_1 0	0 y_1
2	x_2 600	300 y_2
3	x_3 0	300 y_3

Table 2:- Element connecting table.

Elements	1	2	3	→ Local no.
①	1 (x_1,y_1)	2 (x_2,y_2)	3 (x_3,y_3)	& → Global node

A/c to Local nodes.

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Step 2 Element stiff. matrix / Global stiff. m

(because only 1 ele.)

$K^{(e)} = t \cdot A \cdot B^T \cdot D \cdot B$

$A = \frac{1}{2}|J|$

$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} 0 & -300 \\ 600 & 0 \end{bmatrix}$

$J = -1,8,0000$

$A = \frac{1}{2}|180000|$

$A = 90,000 \ mm^2$

$t = 10 \ mm$

$D = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix}$

$= 219.78 \times 10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$

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$= 10^3 \begin{bmatrix} 219.78 & 65.93 & \\ 65.93 & & \\ 0 & & \end{bmatrix}$

$B = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$

$= \frac{1}{|180,000|} \begin{bmatrix} 0 & 0 & 300 & 0 & -300 & 0 \\ 0 & -600 & 0 & 0 & 0 & 600 \\ -600 & 0 & 0 & 300 & 600 & -300 \end{bmatrix}$

$B^T = \frac{1}{1,80,000} \begin{bmatrix} 0 & 0 & -600 \\ 0 & -600 & 0 \\ 300 & 0 & 0 \\ 0 & 0 & 300 \\ -300 & 0 & 600 \\ 0 & 600 & -300 \end{bmatrix}$

$= 6.105 \begin{bmatrix} 0 & 0 & -600 \\ 0 & -600 & 0 \\ 300 & 0 & 0 \\ 0 & 0 & 300 \\ -300 & 0 & 600 \\ 0 & 600 & -300 \end{bmatrix} \times \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$

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$$= 6.105 \begin{bmatrix} 0 & 0 & -210 \\ -180 & -600 & 0 \\ 300 & 90 & 0 \\ 0 & 0 & 2105 \\ -300 & -90 & 210 \\ 180 & 600 & -105 \end{bmatrix}$$

$$= 6.105 \begin{bmatrix} 0 & 0 & -210 \\ -180 & -600 & 0 \\ 300 & 90 & 0 \\ 0 & 0 & 105 \\ -300 & -90 & 210 \\ 180 & 600 & -105 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 300 & 0 & -300 & 0 \\ 0 & -600 & 0 & 0 & 0 & 600 \\ -600 & 0 & 0 & 300 & 600 & -300 \end{bmatrix}$$

	1x	1y	2x	2y	3x	3y	
$= 6.105 \times 10^3$	126	0	0	-63	-126	63	1x
	0	360	-54	0	54	-360	1y
	0	-54	~~360~~ 90	0	-360	54	2x
	-63	0	0	31.5	63	-31.5	2y
	-126	54	-360	63	~~729~~ 216	-117	3x
	63	-360	54	-31.5	-117	391.5	3y

10^3	769.23	0	0	-384.61	-769.23	384.61
	0	2197.8	-329.67	0	329.67	-2197.8
	0	-329.67	2197.8	0	-2197.8	329.67
	-384.61	0	0	192.30	384.61	-192.30
	-769.23	329.67	-2197.8	384.61	1318.68	-714.28
	384	2197.8	329.67	-192.30	-714.28	2390.10

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4) Nodal displ.

$[F] = [K] \times [q]$

$$10^3 \begin{bmatrix} 0 \\ 0 \\ 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} & & & & & \\ & & & & & \\ & & 549.95 \; (~~2197.8~~) & 0 & & \\ & & 0 & 192.30 & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ q_{2x} \\ q_{2y} \\ 0 \\ 0 \end{bmatrix}$$

(columns: 1x, 1y, 2x, 2y, 3x, 3y)

$2197.8 \, q_{2x} + 0 \, q_{2y} = 5$

$0 \, q_{2x} + 192.30 \, q_{2y} = -3$

$q_{2x} = ~~2.27 \times 10^{-3}~~ \; 9.1 \times 10^{-3}$

$q_{2y} = -0.015$

5) Reaction at support

$R_{1x} =$

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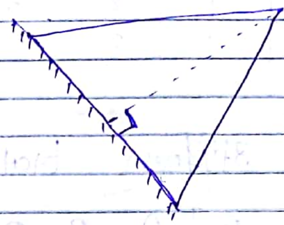
Que] (8) of Q.P pg.4



1) Discritization.

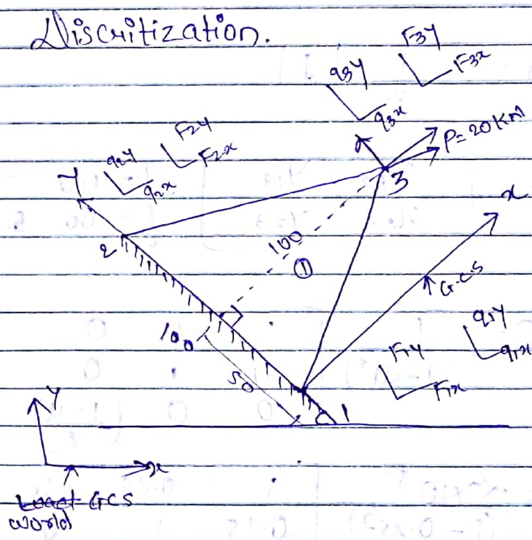


Table ① Nodal Co-ordinate table.

Nodes	x		Y	
1	x_1	0	0	y_1
2	x_2	0	100	y_2
3	x_3	100	50	y_3

Table 2 Element Connecting table

Element	1	2	3
(D)	1	2	3

(5) Element Stiffness matrix

$K^e = t \cdot B^T \cdot A \cdot B \cdot D$

$A = \frac{1}{2}|J|$

$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -100 & -50 \\ -100 & 50 \end{bmatrix}$

$D = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix}$

$= \frac{2\times10^5}{(1-0.25^2)} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$

$= 213.33\times10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$

$B = \frac{1}{|J|} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$

$= \frac{1}{10,000} \begin{bmatrix} 50 & 0 & 50 & 0 & -100 & 0 \\ 0 & 100 & 0 & -100 & 0 & 0 \\ 100 & 50 & -100 & 50 & 0 & -100 \end{bmatrix}$

$A = \frac{1}{2}|10,000|$

$= 5000 \text{ mm}^2$

$K = 8 \times 5000 \times \frac{1}{10,000} \times \begin{bmatrix} 50 & 0 & 100 \\ 0 & 100 & 50 \\ 50 & 0 & -100 \\ 0 & -100 & 50 \\ -100 & 0 & 0 \\ 0 & 0 & -100 \end{bmatrix}^{B^T}$

$\times \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times$

$\begin{bmatrix} 50 & 0 & 50 & 0 & -100 & 0 \\ 0 & 100 & 0 & -100 & 0 & 0 \\ 100 & 50 & -100 & 50 & 0 & -100 \end{bmatrix}$

$$K = 85.332 \begin{bmatrix} 50 & 12.5 & 37.5 \\ 25 & 100 & 18.75 \\ 50 & 12.5 & -37.5 \\ -25 & -100 & 18.75 \\ -100 & -25 & 0 \\ 0 & 0 & -37.5 \end{bmatrix}_{6\times3}$$

$$\times \begin{bmatrix} 50 & 0 & 50 & 0 & -100 & 0 \\ 0 & 100 & 0 & -100 & 0 & 0 \\ 100 & 50 & -100 & 50 & 0 & -100 \end{bmatrix}$$

$$= 85.332 \begin{bmatrix} 6250 & 3125 & -1250 & 625 & -5000 & -3750 \\ -3125 & 109375 & -625 & -9062.5 & -2500 & -1875 \\ -1250 & -625 & 6250 & -3125 & -5000 & 3750 \\ 625 & -9062.5 & -3125 & -2500 & -5312.5 & -468.7 \\ -5000 & -2500 & -5000 & -10625 & -5000 & -234.37 \\ -3750 & -1875 & 3750 & -2500 & -5312.5 & 2343.7 \end{bmatrix}$$

(annotations: 109375, 10000, 0, 0, 3750)

3) Displacement of nodes

$$F = [K] \times [q]$$

85.33

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \\ 0 \end{bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & 10000 & 0 \\ & & & & 0 & 3750 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

$20 \times 10^3 = 1000\, q_{3x} + 0$

$0 = 0 + 3750\, q_{3y}$

$q_{3x} = 2\,mm \qquad q_{3y} = 0$

$$q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} mm$$

Now we cal. q' for world Ces.

$q'_{3x} = 2 \cos 45 = 1.41\ mm$

$q'_{3y} = 2 \sin(45) = 1.41\ mm$

$$q' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.41 \\ 1.41 \end{bmatrix} mm$$

4) Reaction at support.

$R_{1x} = -10\ KN\ (\leftarrow)$

$R_{1y} = -5\ KN\ (\downarrow)$

$R_{2x} = -10\ KN\ (\leftarrow)$

$R_{2y} = 5\ KN\ (\downarrow)$

5). Strain

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = B \cdot q$$

$$\epsilon = \begin{bmatrix} 50 & 0 & 50 & 0 & -100 & 0 \\ 0 & 100 & 0 & -100 & 0 & 0 \\ 100 & 50 & -100 & 50 & 0 & -100 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \times \frac{1}{10000}$$

$$\epsilon = \begin{bmatrix} -200 \\ 0 \\ 0 \end{bmatrix} \times \frac{1}{10000} = \begin{bmatrix} -0.02 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

6) Stress (Hooke's law).

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = D\epsilon = DBq.$$

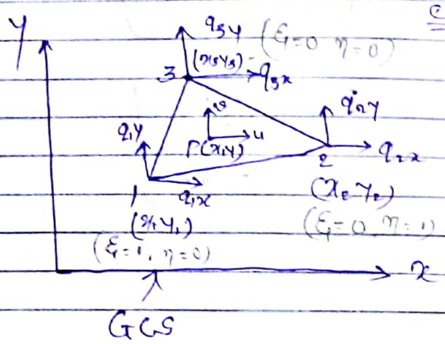
$$\sigma = \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \times \begin{bmatrix} -0.02 \\ 0 \\ 0 \end{bmatrix} \times 215.33 \times 10^3$$

$$\sigma = \begin{bmatrix} -0.02 \\ -0.005 \\ 0 \end{bmatrix} \times 215.33 \times 10^3$$

$$= \begin{bmatrix} -4266.6 \\ -1066.68 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

★ Shape fun for 2-D CST Element or triangular element

(8)



Let q_1x q_1y, q_2x q_2y & q_3x q_3y be the displacements at node 1 2 & 3 respt. in x & y dirn.

Let (x_1, y_1) (x_2, y_2) & (x_3, y_3) be the Co-ordinate at node 1, 2 & 3 respt. in x & y dirn

Let P(x,y) be the any pt. within an element as sh. in figure.

Let (u, v) be the displ. of pt. 'P' in x & y dirn respt. within an element.

Let N_1, N_2 & N_3 be the shape fun represents for CST elements. Or 2-D triangular element.

Let (ξ, η) be the natural Co-ordinates of 2-D CST element.

Hence we can write displacement at pt. P is given by

$$\left.\begin{aligned} u &= N_1 q_1x + N_2 q_2x + N_3 q_3x \\ v &= N_1 q_1y + N_2 q_2y + N_3 q_3y \end{aligned}\right\} \quad (1)$$

$$\text{Let} \quad \left.\begin{aligned} N_1 &= \xi \\ N_2 &= \eta \\ N_3 &= 1 - \xi - \eta \end{aligned}\right\} \quad (2)$$

Using eqn (2) in (1) so that the displ. at pt. 'P' to be represent in terms of natural Co-ordinates.

$$u = \xi q_1x + \eta q_2x + (1 - \xi - \eta) q_3x$$

$$u = \xi(q_1x - q_3x) + \eta(q_2x - q_3x) + q_3x$$

$$v = \xi q_1y + \eta q_2y + (1 - \xi - \eta) q_3y$$

$$v = \xi(q_1y - q_3y) + \eta(q_2y - q_3y) + q_3y \quad (3)$$

Similarly we can write

$$\left.\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned}\right\} \quad (4)$$

Using eq^n (2) in (1) so that the co-ordinates of pt. P to be represent in terms of natural Co-ordinates.

$$x = \xi x_1 + \eta x_2 + (1-\xi-\eta) x_3$$

$$x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$$

$$y = \xi(y_1 - y_3) + \eta(y_2 - y_3) + y_3 \quad \text{—(5)}$$

Now represent natural co-ordinates for node 1 2 3 respt as below.

We know that

$$N_1 + N_2 + N_3 = 1 \quad \text{—(6)}$$

We know that

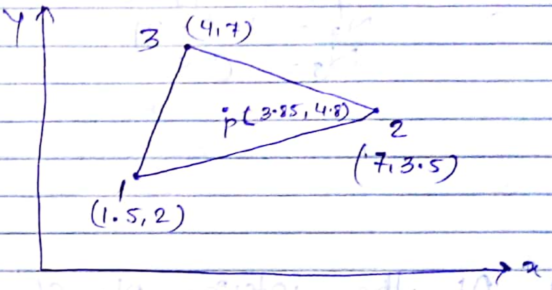
At node 1, $N_1 = 1$, $N_2 = N_3 = 0$

At node 2, $N_2 = 1$, N_1 & $N_3 = 0$

At node 3, $N_3 = 1$, $N_1 = N_2 = 0$

at node 1, $N_1 = \xi = 1$, $N_2 = \eta = 0$
at node 2, $N_2 = \eta = 1$, $N_1 = \xi = 0$
at node 3, $N_1 = \xi = 0$, $N_2 = \eta = 0$

Que] Evaluate a shape fn N_1, N_2 & N_3 at pt. 'P' for the Δ element sh. in fig.



$x = 3.85$
$y = 4.8$
$x_1 = 1.5$ $y_1 = 2$
$x_2 = 7$ $y_2 = 5$
$x_3 = 4$ $y_3 = 7$.

$$x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$$
$$y = \xi(y_1 - y_3) + \eta(y_2 - y_3) + y_3$$ from (1)

$$x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$$

$$3.85 = \xi(1.5 - 4) + \eta(7 - 4) + 4$$
$$3.85 = -2.5\xi + 3\eta + 4$$
$$-0.15 =$$

$$4.8 = \xi(2 - 7) + \eta(5 - 7) + 7$$
$$4.8 = -5\xi + 2\eta + 7$$

$= 2.2 =$

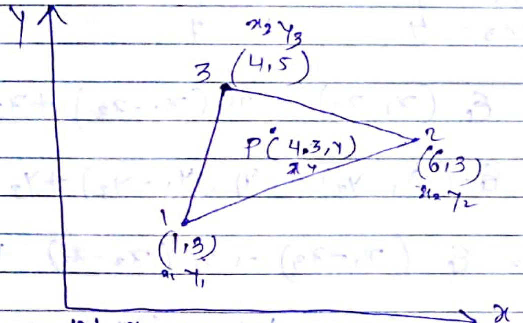
$\xi =$ ~~[illegible]~~ 0.3

$\eta =$ ~~[illegible]~~ 0.2

$N_3 = 1 - \xi - \eta$

$N_3 = 0.5$

Q) At the interior pt. of Δ element the x-coordinate of pt. P is 4.3 & $N_1 = 0.43$. Determine N_2, N_3 & y-coordinate at pt. P



$x = N_1 x_1 + N_2 x_2 + N_3 x_3$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3$

$x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$

$y = \xi(y_1 - y_3) + \eta(y_2 - y_3) + y_3$

$4.3 = 0.3(1-4) + \eta(6-4) + 4$

$y = 0.3(3-5) + \eta(3-5) + 5$

$\eta = 0.6 = N_2$

$y = 3.2$

$N_1 + N_2 + N_3 = 1$

$0.3 + 0.6 + N_3 = 1$

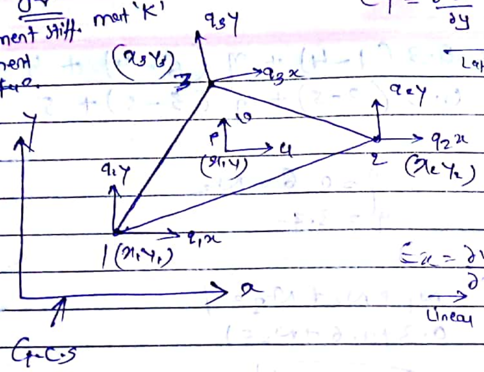
$N_3 = 0.1$

Q] Derive strain displ. relation mat. 'B' for CST element using shape fn? ↑ Linear

M.Imp 10 or

Derive element stiff. mat 'K' for CST element using shape fn.

$\epsilon_y = \frac{\partial v}{\partial y}$



Lateral $\frac{\partial v}{\partial x}$

$\epsilon_x = \frac{\partial u}{\partial x}$ Linear

Lateral ↑ $\frac{\partial u}{\partial y}$

The strain at any pt. within CST element is given by.

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}_{3\times 1}$$

$$\epsilon = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}_{3\times 1} \quad \longleftarrow (1)$$

differentiating displ. 'u' at any pt. by the natural coordinates (ξ, η) using chain differentiation rule we can write.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

Matrix Eq^n.

$$\begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix}_{2\times 1} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}_{2\times 2} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \partial u/\partial y \end{bmatrix}_{2\times 1} \longrightarrow (2)$$

⇓

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}_{2\times 2}$$

J → Jacobian matrix

differentiating

$$J = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}_{2\times 2}$$

Note:→ The jacobian mat. is used to get the area of CST or Δ angular element & is given by.

$$A = \frac{1}{2}|J|$$

$$= \frac{1}{2}\left| x_{13} \cdot y_{23} - y_{13} \cdot x_{23} \right|$$

Eqn ① can be written as

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}_{2\times1} = J^{-1}_{2\times2} \begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix}_{2\times1} \quad \text{③}$$

Where J^{-1} is the inverse of jacobian matrix & is given by

$$J^{-1} = \frac{Adj\,[J]}{|J|} = \frac{1}{|J|}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}_{2\times2}$$

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}_{2\times1} = \frac{1}{|J|}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}_{2\times2} \begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix}_{2\times1} \quad \text{④}$$

where $u = (q_{1x} - q_{3x}) \cdot \xi + (q_{2x} - q_{3x}) \cdot \eta + q_{3x}$

differentiation ④

$$\text{④} \Rightarrow \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \frac{1}{|J|}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}_{2\times2} \begin{bmatrix} q_{1x} - q_{3x} \\ q_{2x} - q_{3x} \end{bmatrix}_{2\times1}$$

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}_{2\times1} = \frac{1}{|J|}\begin{bmatrix} y_{23}(q_{1x} - q_{3x}) - y_{13}(q_{2x} - q_{3x}) \\ -x_{23}(q_{1x} - q_{3x}) + x_{13}(q_{2x} - q_{3x}) \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{|J|}\left[q_{1x} \cdot y_{23} + q_{2x} \cdot y_{31} + q_{3x} \cdot y_{12} \right] \quad \text{⑤}$$

$-y_{23}\, q_{3x} + y_{13} \cdot q_{3x}$

$q_{3x}\,(y_{13} - y_{23})$ — simplifying

$(y_1 - y_3 - y_2 + y_3)$

$(q_{3x}\, y_{12})$

$$\frac{\partial u}{\partial y} = \frac{1}{|J|}\left[q_{1x} \cdot x_{32} + q_{2x} \cdot x_{13} + q_{3x} \cdot x_{21} \right] \quad \text{⑥}$$

$x_{23} \cdot q_{3x} - x_{13} \cdot q_{3x}$

$q_{3x}\,(x_2 - x_3 - x_1 + x_3)$

$(q_{3x}\, x_{21})$

Now differentiating displacement 'v' w.r.t. natural coordinate ξ & η.
Similarly after analysing the eq^n we can write

$$\frac{\partial v}{\partial x} = \frac{1}{|J|}\left[q_{1y} \cdot y_{23} + q_{2y} \cdot y_{31} + q_{3y} \cdot y_{12}\right] \quad \text{--- (7)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{|J|}\left[q_{1y} \cdot x_{32} + q_{2y} \cdot x_{13} + q_{3y} \cdot x_{21}\right] \quad \text{--- (8)}$$

using eq^n (5) (6) (7) & (8) in eq^n (1)

(1) ⇒
$$\epsilon = \frac{1}{|J|}\begin{bmatrix} q_{1x} \cdot y_{23} + q_{2x} \cdot y_{31} + q_{3x} \cdot y_{12} \\ q_{1y} \cdot x_{32} + q_{2y} \cdot x_{13} + q_{3y} \cdot x_{21} \\ \left(q_{1x} \cdot x_{32} + q_{2x} \cdot x_{13} + q_{3x} \cdot x_{21} + q_{1y} \cdot y_{23} + q_{2y} \cdot y_{31} + q_{3y} \cdot y_{12}\right) \end{bmatrix}_{3\times1}$$

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}_{3\times1} = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times6} \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}_{6\times1}$$

$$\epsilon_{3\times1} = B_{3\times6} \cdot q_{6\times1} \quad \text{--- (9)}$$

Where

Strain displ. relation mat.
$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times6}$$

Element Stiffness matrix for CST Element

S.E at a pt / volume $= \frac{1}{2} \cdot \sigma^T \cdot \epsilon$

SE at a pt $= \frac{1}{2} \sigma^T \cdot \epsilon \cdot$ Volume at pt

SE at pt $= \frac{1}{2} \sigma^T \cdot \epsilon \cdot t \cdot dA$ --- (10)

$[\because V = t \cdot dA]$

∴ Total SE for CST element

$$\int_0^A \frac{1}{2} \sigma^T \cdot \epsilon \cdot t \cdot dA$$

$$= \frac{1}{2} \sigma^T \epsilon \cdot t \cdot [A]_0^A$$

$$= \frac{1}{2} \sigma^T \cdot \epsilon \cdot t \cdot A$$

$$= \frac{1}{2} (D \cdot \epsilon)^T \cdot \epsilon \cdot t \cdot A$$

$$= \frac{1}{2} \epsilon^T \cdot D^T \cdot \epsilon \cdot t \cdot A$$

$\sigma = D \cdot \epsilon$
$\sigma^T = (D \cdot \epsilon)^T = \epsilon^T \cdot D^T$
$(a \cdot b)^T = b^T \cdot a^T$
$\because \epsilon = B \cdot q$

Scanned by CamScanner

$$TSE = \frac{1}{2}(B \cdot q)^T \cdot D^T (B \cdot q) \cdot t \cdot A$$

$$= \frac{1}{2} q^T B^T \cdot D^T \cdot B \cdot q \cdot t \cdot A$$

$$= \frac{1}{2} \underset{1\times6}{q^T} \, t \cdot A \cdot \underset{6\times3}{B^T} \underset{3\times3}{D^T} \cdot \underset{3\times6}{B} \cdot \underset{6\times1}{q}$$

$$TSE = \frac{1}{2} q^T \cdot K^{(e)}_{6\times6} \cdot q_{6\times1} \quad \text{--- (11)}$$

Where

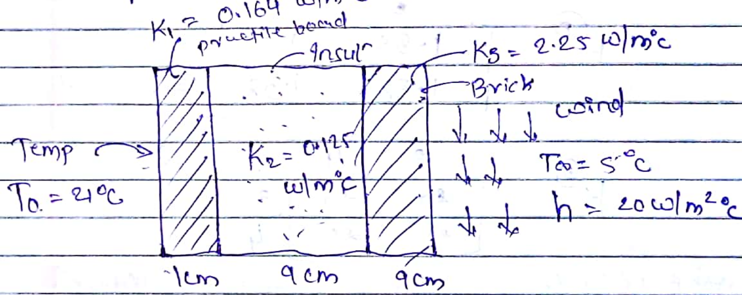
$$K_{6\times6} = t \cdot A \cdot \underset{6\times3}{B^T} \cdot \underset{3\times3}{D^T} \cdot \underset{3\times6}{B}$$

$$K^{(e)}_{6\times6} = t \cdot A \cdot B^T \cdot D \cdot B \qquad \because D^T = D$$

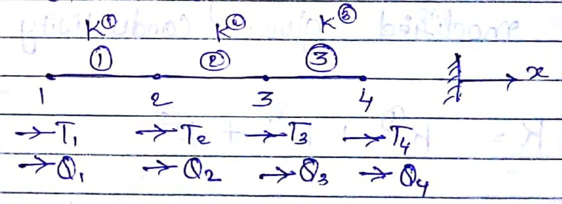
Unit IV

15m

Q) The interior wall of room is maintain at the temp of 21°C. The wall is built using partial load insulation and brick as sh. on a metal day the outside air temp has convection coeff. 2014 Determine temp at wall interfaces and the rate of heat loss from the room per m^2.



⇒ 1) Discritization



Where $K^{(1)}, K^{(2)}, K^{(3)}, K^{(4)}$ → Th. conductivity

T_1, T_2, T_3, T_4 → nodal wall temp.

Q_1, Q_2, Q_3, Q_4 → Heat flux at node 1, 2, 3, 4 resp.

Assuming Area $A = 1m^2$

always but depends on unit of k. (If not given)

1 cm = 1/10^-2 m ÷ 100 m it

2] Element Conductivity matrix

$$K^{(e)} = \frac{KA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{(1)} = \begin{bmatrix} 16.4 & -16.4 \\ -16.4 & 16.4 \end{bmatrix} \; W/°C$$ (DOF 1, 2)

$$K^{(2)} = \begin{bmatrix} 1.38 & -1.38 \\ -1.38 & 1.38 \end{bmatrix}$$ (DOF 2, 3)

$$K^{(3)} = \begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix}$$ (DOF 3, 4)

3) Global conductivity matrix & modified global conductivity mat.

$$K = K^{(1)} + K^{(2)} + K^{(3)}$$

	1	2	3	4	
=	16.4	16.4	0	0	1
	-16.4	17.78	-1.38	0	2
	0	-1.38	26.38	25	3
	0	0	-25	25	4

4×4

3] As convection h i.e 20 W/m²°C is mention at node 4 hence convection B.C i.e $h \cdot A = 20 \times 1 = 20 \; W/°C$ is added to 4×4 location of global stiff. mat K'

So modified K

	1	2	3	4	
K =	16.4	16.4	0	0	1
	-16.4	17.78	-1.38	0	2
	0	-1.38	26.38	25	3
	0	0	-25	45	4

4] Nodal temp / temp at wall interface

A/c to PMPE

$[Q] = [K] \cdot [T]$ ——— ★

global heat flux; Global Conductivity matrix; global temp. matrix

using B.C

1) Temp B.C, i.e $T_1 = 21°C$, $T_2, T_3 \& T_4 = ?$

2] Convection B.C / Heat flux B.C

$Q_4 = h \cdot A \cdot T_\infty$

$= 20 \times 1 \times 5$

$Q_4 = 100$ watts.

$$Q_2 = \frac{K_1 T_0}{L_1} = \frac{0.164 \times 21}{0.01}$$

$$Q_2 = 344.4 \text{ watts}$$

$$[Q] = [K][T]$$

$$\begin{bmatrix} Q_1 \\ 344.4 \\ 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 16.4 & -16.4 & 0 & 0 \\ -16.4 & 17.78 & -1.38 & 0 \\ 0 & -1.38 & 26.38 & 25 \\ 0 & 0 & -25 & 45 \end{bmatrix} \begin{bmatrix} 21 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

use elimination approach
(Temp is known that why 1st row 1st col. cut) Q ki 1st node ka diya hai

$$17.78\, T_2 - 1.38\, T_3 + 0 = 344.4$$
$$-1.38\, T_2 + 26.38\, T_3 + 25 T_4 = 0$$
$$0\, T_2 - 25\, T_3 + 45\, T_4 = 100$$

$T_2 = 19.31$
$T_3 = 6.65$
$T_4 = 5.91$

Ans Temp field = $\begin{bmatrix} 21 \\ 19.31 \\ 6.65 \\ 5.91 \end{bmatrix}$

- Rate of heat loss i.e heat flux at entry & exist i.e Q_1 & Q_4

H.W Q.Pap [W 13 Q9, W14 Q9]

Using Global K eqn. x 2 [1st row x 1st col.]

$$Q_1 = 16.4 \times 21 - 16.4 \times T_2 (19.31)$$

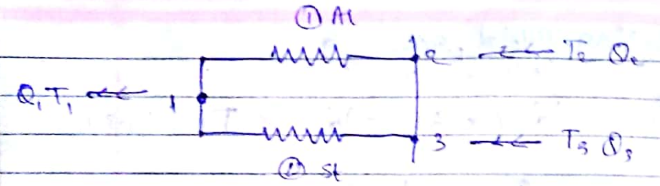
$Q_1 = 18.37$ watts

$Q_4 = 100$ watts.

Heat loss $= Q_4 - Q_1$
$= 81.63$ watt

Q] A steel shaft and al. tube are connected to a fixed support and a rigid disc as sh. in fig. If the torque applied at the en (Q.P. last)

→ i) Discritizing the component into 2 finite element & representing into the form of line & 1D spring element as sh. in fig.



$A_1 = \frac{\pi}{4}(d_o^2 - d_i^2) = 1608.5\ mm$

$A_2 = \frac{\pi}{4} d^2 = 1963.5\ mm$

2) Element Stiff mat

$K^{(1)} = \frac{G_1 J_1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$J_1 = \frac{\pi}{32}(d_o^4 - d_i^4) = 1.672 \times 10^6\ mm^4$

$J_2 = \frac{\pi}{32} d^4 = 0.613 \times 10^6\ mm^4$

$K^{(1)} = \frac{27 \times 10^3 \times 1.672 \times 10^6}{500}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= 10^6 \begin{bmatrix} 90.28 & -90.28 \\ -90.28 & 90.28 \end{bmatrix}$ (rows/cols 1, 2)

$K^{(2)} = \frac{77 \times 10^3 \times 0.613 \times 10^6}{500}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$K^{(2)} = 10^6 \begin{bmatrix} 94.40 & -94.40 \\ -94.40 & 94.40 \end{bmatrix}$ (rows/cols 1, 3)

3] Global stiff mat.

$K = 10^6 \begin{bmatrix} 184.68 & -90.28 & -94.40 \\ -90.28 & 90.28 & 0 \\ -94.40 & 0 & 94.40 \end{bmatrix}$ (rows/cols 1, 2, 3)

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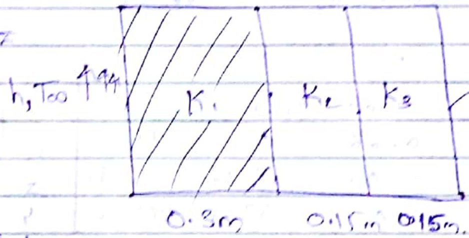
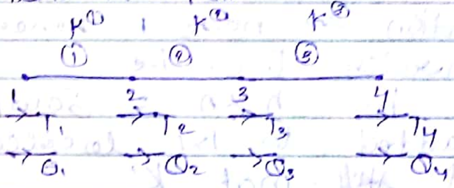


$T_0 = 20°C$
$K_1 = 20\ W/m°C$
$K_2 = 30\ W/m°C$
$K_3 = 50\ W/m°C$
$T_\infty = 800°C$

$q_1 = h \cdot T_\infty = 4 \times 800$

⇒ 1) Discritization



2) Element Conductivity table.

$$K^{(1)} = \frac{KA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

DOF: 1, 2

$$= \begin{bmatrix} 66.66 & -66.66 \\ -66.66 & 66.66 \end{bmatrix}$$

$$K^{(2)} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$
(columns 2, 3 ← DOF)

$$K^{(3)} = \begin{bmatrix} 333.33 & -333.33 \\ -333.33 & 333.33 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$
(columns 3, 4 ← DOF)

3) Global condn. mat. & modified global conductivity matrix.

$$K = \begin{bmatrix} 66.66 & -66.66 & 0 & 0 \\ -66.66 & 66.66 & 0 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$
(columns 1, 2, 3, 4)

As convection h i.e 30 W/m²°C is mention at node 1 hence convection B.C i.e

$hA = h \times A = 30 \text{ W/m}^2\text{C}$

is added to 1×1 location of global stiff. mat 'K'

$$K = \begin{bmatrix} 96.66 & -66.66 & 0 & 0 \\ -66.66 & 66.66 & 0 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix}$$

4) Nodal temp.

$[Q] = [F][T]$.

1) Temp B.C i.e $T_4 = 20°C$ T_1, T_2 & $T_3 = ?$

2) Convection B.C

$Q_1 = h A T_\infty$

$= 30 \times 1 \times 800$

$Q_1 = 24000$ watts

$Q_3 = \dfrac{K_3 T_0}{L_3}$

$Q_3 = 6.66 \times 10^3$ watt

$[Q] = [K][T]$

$$10^3 \begin{Bmatrix} 24 \\ 0 \\ 6.66 \\ \cancel{Q_4} \end{Bmatrix} = \begin{bmatrix} 96.66 & -66.66 & 0 & 0 \\ -66.66 & 66.66 & 0 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ \cancel{0} & \cancel{0} & \cancel{-333.33} & \cancel{333.33} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \cancel{20} \end{Bmatrix}$$

$96.66\, T_1 - 66.66\, T_2 + 0 = 24 \times 10^3$

$-66.66\, T_1 + 66.66\, T_2 + 0 = 0$

$0 - 200\, T_2 + 533.33\, T_3 = 6.66 \times 10^3$

$T_1 = 800$

$T_2 = 800$

$T_3 = 312.50$

$T_4 = 20$.

H.L = -93.54 × ... watt

- Rate of heat loss i.e at entry & exist. Q_1 & Q_4

$Q_4 = -333.33 \times 800 + 533.33 \times 312.50 \; 20$

List of Derivation

any 4 in paper 100%

1) Element stiff. mat. for 1-D bar or line element using direct or conventional approach. (2-noded) for line also, or for bar also

2) Principle of min. potential energy.

3) Shape fun for 1-D bar element / line element (2 noded) / Lagrange's shape fun (theory)

4) Shape fun for 1-D quadratic bar element (3-noded)

5) Element stiff. mat. for bar or line element using shape fun (2 noded)

6) Element stiff mat for truss element (4×4) along with stresses & strain in the truss element.

7) Hermit shape fun for Beam element. (2 node)

8) Shape fun for CST elem. or Δ element

9) Plane stress plain strain condn.

10) Strain displ. relation matrix B'

or

Element stiff. mat. 'k' for CST element (6×6) using shape fun.

Unit: 3 (Part)

Multi point Constraint

In this pb. where inclined rollers or rigid connection are to be model, the B.C take the form

$$B_1 q_1 + B_2 q_2 = B_0$$

where B_0, B_1 & B_2 are kn. as const. & q_1 & q_2 be the displ. at that node. Such B.C are reffered as multipoint constraint in the literature. The penaulty approach is apply for this situation.

In this type of pb. the global stiff. & force mat are to be modified as below.

K → 1 2 3 4 ← DOF

Global Stiff. matrix: K = [ ] (rows 1, 2, 3, 4)

* Select 'C' from global stiff. matrix which is kn. as large no. of stiff. matrix so that the potensial energy eqn is to be minimum.

$$C = [\text{Max value of } K] \times 10^4$$

Now modify stiffness & force mat. as below

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(columns 1, 2)

modified K:

$$K = \begin{bmatrix} K_{11} + C B_1^2 & K_{12} + C B_1 B_2 \\ K_{21} + C B_1 B_2 & K_{22} + C B_2^2 \end{bmatrix}$$

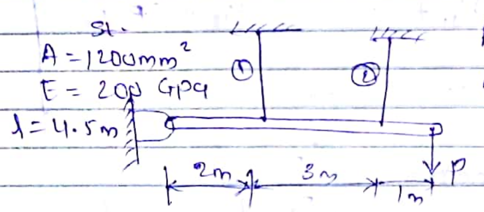
& force matrix $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$

modified $\begin{bmatrix} F_1 + C B_0 B_1 \\ F_2 + C B_0 B_2 \end{bmatrix}$

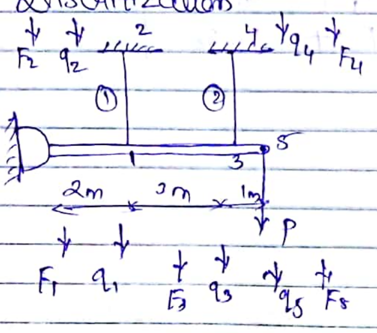
Q] Consider the Structure sh. in fig. A rigid bar of negligible mass, pinned at one end, is supported by steel rod & Al. rod. A load 'P' = 30 kN is applied as sh.

1) Model the structure using 2 finite elements. What are the B.C of your model.

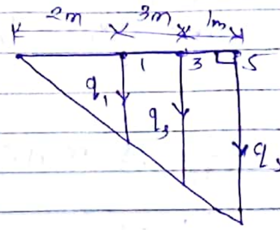
2] Develop the modified stiff. mat. & modified load vector. Solve the eq^n for (Q) then determine element stresses.



⇒ 1] Discritization



displ. diag.



using similarity of triangle.

$$\frac{q_1}{q_5} = \frac{2}{6} \rightarrow \text{length from origin}$$

$q_1 = 0.333 q_5$

$q_1 - 0.333 q_5 = 0$ —— ①

$$\frac{q_3}{q_5} = \frac{5}{6}$$

$q_3 = 0.83 q_5$

$q_3 - 0.83 q_5 = 0$ —— ②

Boundary cond^n at node 2 & 4 are obvious i.e $Q_2 = Q_4 = 0$

Now since the rigid bar has to remain straight Q_1, Q_3 & Q_5

are related as sh. in fig. The multipt. Constraint due to the rigid bar configuration are given by eqⁿ (1) & (2)

Step 2:→ Element Stiffness matrix

$$K^{(e)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{(1)} = \frac{1200 \times 200 \times 10^3}{4.5 \times 10^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 53.33 & -53.33 \\ -53.33 & 53.33 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

(DOF 1, 2)

$$K^{(2)} = 10^3 \begin{bmatrix} 21 & -21 \\ -21 & 21 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

(DOF 3, 4)

Step 3 → Global stiff. mat. & modified global stiff. mat.

$$K = 10^3 \begin{bmatrix} 53.33 & -53.33 & 0 & 0 & 0 \\ -53.33 & 53.33 & 0 & 0 & 0 \\ 0 & 0 & 21 & -21 & 0 \\ 0 & 0 & -21 & 21 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

(columns 1 2 3 4 5)

$$C = [53.33 \times 10^3] \times 10^4$$

Where C → Large no. of global stiff mat.

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \longrightarrow \begin{bmatrix} K_{11} + CB_1^2 & K_{12} + CB_1B_2 \\ K_{21} + CB_1B_2 & K_{22} + CB_2^2 \end{bmatrix}$$

$$q_1 - 0.333 q_5 = 0$$

$$B_1 = 1, \quad B_2 = -0.333, \quad B_0 = 0$$

$$K = 10^3 \begin{bmatrix} 53.33 \times 10^4 & -17.59 \times 10^4 \\ -17.59 \times 10^4 & 5.80 \times 10^4 \end{bmatrix} \begin{matrix} 1 \\ 5 \end{matrix}$$

(DOF 1, 5)

$$q_3 - 0.83 q_5 = 0.$$

$$B_1 = 1, \quad B_2 = -0.83, \quad B_0 = 0$$

$$K = 10^3 \begin{bmatrix} 53.33 \times 10^4 & -44.26 \times 10^4 \\ -44.26 \times 10^4 & 36.73 \times 10^4 \end{bmatrix} \begin{matrix} 3 \\ 5 \end{matrix}$$

(DOF 3, 5)

Now add both multipt. constraint stiff. in original stiff. mat. gives you a modified stiff. mat. / final S.M

$$K = 10^3 \begin{bmatrix} 53.33\times10^4 & -53.33 & 0 & 0 & -17.59\times10^4 \\ -53.33 & 53.33 & 0 & 0 & 0 \\ 0 & 0 & 53.33\times10^4 & -21 & -44.26\times10^4 \\ 0 & 0 & -21 & 21 & 0 \\ -17.5\times10^4 & 0 & -44.26\times10^4 & 0 & 42.53\times10^4 \end{bmatrix}$$

(rows and columns numbered 1 to 5)

Step 4 → Global force mat. & modified global force mat.

$$F = 10^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

mod.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_1 + C\beta_0\beta_1 \\ F_2 + C\beta_0\beta_2 \end{bmatrix}$$

as $\beta_0 = 0$ in both eqn hence this remain same

Step 5 → Nodal displ. field.

A/c to PMPE

$$[F] = [K] \times [q]$$

$$10^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix} = 10^3 \begin{bmatrix} 53.33\times10^4 & 0 & -17.59\times10^4 \\ 0 & 53.33\times10^4 & -44.26\times10^4 \\ -17.5\times10^4 & -44.26\times10^4 & 42.53\times10^4 \end{bmatrix} \begin{bmatrix} q_1 \\ 0 \\ q_3 \\ 0 \\ q_5 \end{bmatrix}$$

(columns 1, 3, 5)

$$53.33\times10^4 q_1 + 0 q_3 + 17.59\times10^4 q_5 = 0$$
$$0 q_1 + 53.33\times10^4 q_3 - 44.26\times10^4 q_5 = 0$$
$$-17.5\times10^4 q_1 + 44.26\times10^4 q_3 + 42.53\times10^4 q_5 = 30$$

$q_1 = 0.039$

$q_3 = 0.0981$

$q_5 = 0.118$

Ans

$$q = \begin{bmatrix} 0.039 \\ 0 \\ 0.0981 \\ 0 \\ 0.118 \end{bmatrix} \text{mm}$$

Step 6 → Stress in element.

$$\sigma^{(1)} = \frac{E}{L}[u_1 - u_2]$$

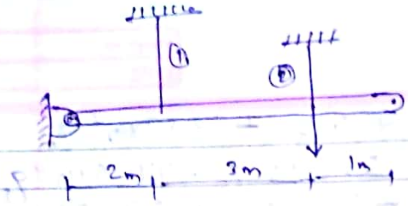
$$= \frac{200\times10^3}{4.5\times10^3}[0.039 - 0]$$

$$\sigma^{(1)} = 1.73 \text{ N/mm}^2$$

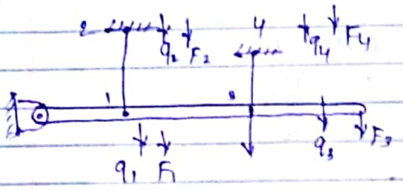
$$\sigma^{(2)} = \frac{E}{L}[u_3 - u_4]$$

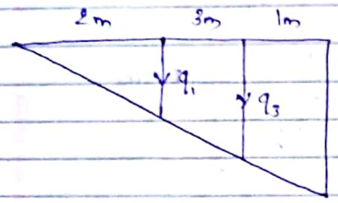
$$\sigma^{(2)} = 2.28 \text{ (MPa)}$$

Q]



⇒ 1] Discretization





Law of similar Δ

$$\frac{q_1}{q_3} = \frac{2}{3} = 0.4$$

$$\boxed{q_1 - 0.4 q_3 = 0} \quad \text{——— (1)}$$

B.C at node (2) & (4) are obvious ∴ $q_2 = q_4 = 0$

2) Element Stiffness mat.

$$K^{(e)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{(1)} = \frac{1200 \times 200 \times 10^3}{4.5 \times 10^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3\begin{bmatrix} 53.33 & -53.33 \\ -53.33 & 53.33 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$K^{(2)} = 10^3\begin{bmatrix} 21 & -21 \\ -21 & 21 \end{bmatrix}\begin{matrix} 3 \\ 4 \end{matrix}$$

3] Global Stiff. mat.

$$K = K^{(1)} + K^{(2)}$$

$$K = 10^3\begin{bmatrix} 53.33 & -53.33 & 0 & 0 \\ -53.33 & 53.33 & 0 & 0 \\ 0 & 0 & 21 & -21 \\ 0 & 0 & -21 & 21 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Where $C = (53.33 \times 10^3) \times 10^4$

$B_1 = 1, \quad B_2 = -0.4, \quad B_0 = 0$

$$K = \begin{bmatrix} CB_1^2 & CB_1B_2 \\ CB_1B_2 & CB_2^2 \end{bmatrix} = \begin{bmatrix} 53.33 \times 10^4 & -21.33 \times 10^4 \\ -21.33 \times 10^4 & 8.53 \times 10^4 \end{bmatrix}\begin{matrix} 1 \\ 3 \end{matrix} \times 10^3$$

Modified global Stiffness matrix

$$K = 10^3\begin{bmatrix} 53.33 \times 10^4 & -53.33 & -21.33 \times 10^4 & 0 \\ -53.33 & 53.33 & 0 & 0 \\ -21.33 \times 10^4 & 0 & 8.53 \times 10^4 & -21 \\ 0 & 0 & -21 & 21 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

4) Global force matrix

$$F = \begin{bmatrix} 0 \\ 0 \\ 30 \\ 0 \end{bmatrix} \times 10^3$$

5) Nodal displ.

As per PMPE

$$10^3 \begin{bmatrix} 0 \\ 0 \\ 30 \\ 0 \end{bmatrix} = 10^3 \begin{bmatrix} 53.33\times10^4 & -21.33\times10^4 \\ -21.33\times10^4 & 8.53\times10^4 \end{bmatrix} \begin{bmatrix} q_1 \\ 0 \\ q_3 \\ 0 \end{bmatrix}$$

$0 = 53.33\times10^4 q_1 - 21.33\times10^4 q_3$

$30 = -21.33\times10^4 q_1 + 8.53\times10^4 q_3$

$q_1 = 0.167$

$q_3 = 0.43$

6) Stress

$\sigma^{(1)} = \frac{E}{L}(u_1 - u_2)$

$= \frac{200\times10^3}{4.5\times10^3}(0.167 - 0)$

$= 7.42 \; (T)$

$\sigma^{(2)} = \frac{E}{L}(u_2 - u_4)$

$= \frac{70\times10^3}{3\times10^3}(0.43 - 0)$

$= 10.33$ mpa (T)

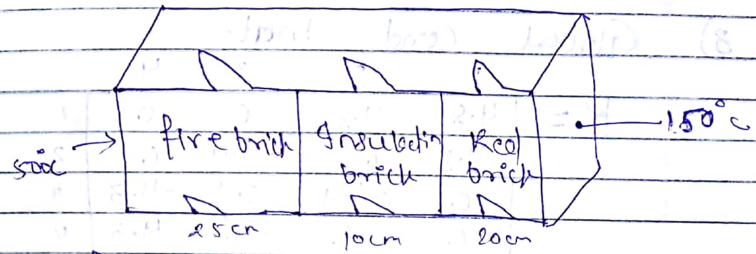
Also calculate the maxm displ. of fulcrum pin at its free end.

$\frac{q_1}{\Delta max} = \frac{2}{5} = 0.33$

$\Delta max = \frac{q_1}{0.33} = 0.501$ mm

$\Delta max = 0.501$ mm

Que] The furnance wall sh. in fig Consist of 25 cm of fire brick ($K_1 = 0.012$ w/cm°c], 10 cm of insulation brick [$K_2 = 0.0014$ w/cm°c] & 20 cm of red brick. $K_3 = 0.0086$ The specify inner & outter temp are 500°c & 150°c respt. Det. the internal temp distribution.



1) Discritization

$\text{(1)} \quad \text{(2)} \quad \text{(3)}$

1 $K^{(1)}$ 2 $K^{(2)}$ 3 $K^{(3)}$ 4

$\rightarrow T_1 \quad \rightarrow T_2 \quad \rightarrow T_3 \quad \rightarrow T_4$

$\rightarrow Q_1 \quad \rightarrow Q_2 \quad \rightarrow Q_3 \quad \rightarrow Q_4$

2) Element ~~St~~ cond. mat.

$$K^{(1)} = \frac{KA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = 1m^2 = 1\times(10^2)^2 = 1\times10^4 \text{ cm}^2$$

$$K^{(1)} = \begin{bmatrix} 4.8 & -4.8 \\ -4.8 & 4.8 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$
(columns 1, 2)

$$K^{(2)} = \begin{bmatrix} 1.4 & -1.4 \\ -1.4 & 1.4 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$
(columns 2, 3)

$$K^{(3)} = \begin{bmatrix} 4.3 & -4.3 \\ -4.3 & 4.3 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$
(columns 3, 4)

3) Global cond. mat.

$$K = \begin{bmatrix} 4.8 & -4.8 & 0 & 0 \\ -4.8 & 6.2 & -1.4 & 0 \\ 0 & -1.4 & 5.7 & -4.3 \\ 0 & 0 & -4.3 & 4.3 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$
(columns 1, 2, 3, 4)

4) Nodal cond.

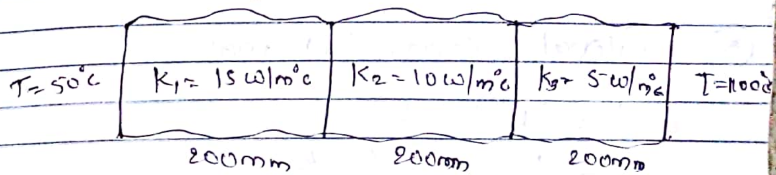
$$[Q] = [K][T]$$

$Q_2 = \frac{TK_1}{L_1} = 0.24$

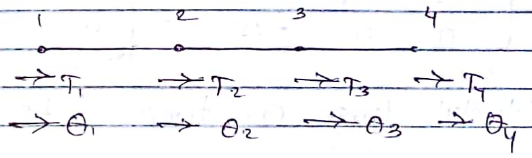
$L_1 \rightarrow$ unit kya lena padta

$Q_3 = \frac{TK_3}{L_3} = 0.0645$

☆ Analyse the temp. distribution through the composite wall as sh. in fig using FEM. also determine 1) the heat entering & existing the system, assume suitable data necessary.



⇒ 1) Discritization



Assuming A = $1mm^2$

2) Element conductivity mat.

$$K^{(1)} = \frac{KA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{15 \times 10^6}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^3 \begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \; w/°c$$
(columns 1, 2)

$$K^{(2)} = 10^3 \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \; W/^\circ C$$

(columns 2, 3)

$$K^{(3)} = 10^3 \times \begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \; W/^\circ C$$

(columns 3, 4)

(5) Global conductivity mat.

$$K = K^{(1)} + K^{(2)} + K^{(3)}$$

$$= \begin{bmatrix} 75 & -75 & 0 & 0 \\ -75 & 125 & -50 & 0 \\ 0 & -50 & 75 & -25 \\ 0 & 0 & -25 & 25 \end{bmatrix}_{4\times4} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(columns 1, 2, 3, 4)

(4) Nodal temp. or temp. distⁿ

A/c to PMPE

$$[Q]_{4\times1} = [K]_{4\times4} [T]_{4\times1}$$

using B.C

$$Q_2 = \frac{K_1 T_0}{L_1} = \frac{15\times50}{200} \; W/m$$

$$Q_2 = 3.75$$

3750

A Plane Stress & Plane Strain Condⁿ

(a)

or

Derive stress transformation mat. 'e' & strain transfⁿ mat 'D' for CST element.

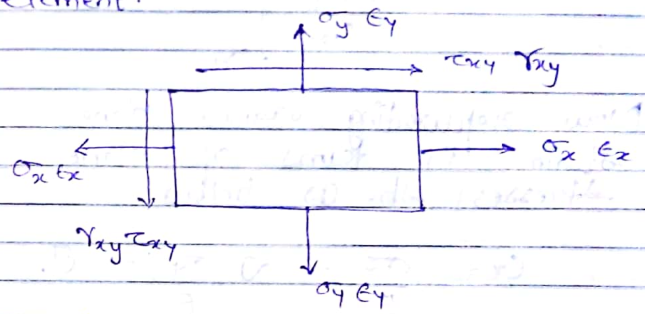


Fig:- Plain Stress & Strain Condⁿ

Let σ_x & σ_y be the normal stresses in x & y dirⁿ resp.
τ_{xy} be the shear stress in xy plane.

Hence plane stress matrix is given by

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{3\times1}$$

Let ϵ_x & ϵ_y be the linear strength in xy plane

Hence plane strain matrix is given by

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}_{3\times1}$$

Now representing various plane strain in terms of plane stresses sh. as below.

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \quad \text{--- (1)}$$

ν = poission ratio

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \quad \text{--- (2)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

But $G = \dfrac{E}{2(1+\nu)}$

$$\therefore \gamma_{xy} = \frac{2(1+\nu)}{E} \cdot \tau_{xy} \quad \text{--- (3)}$$

Matrix eqn of 1, 2 & 3

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\epsilon = C \cdot \sigma \quad \text{--- (4)}$$

where $C = \dfrac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$

C — Stress transfn mat.

Now, representing various plane stresses in terms of plane strain

$$(1) \times \nu \Rightarrow \nu\epsilon_x = \nu\frac{\sigma_x}{E} - \nu^2\frac{\sigma_y}{E}$$

$$(2) \Rightarrow \epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

+

$$\nu\epsilon_x + \epsilon_y = \frac{\sigma_y}{E}(1-\nu^2)$$

$$\therefore \sigma_y = \frac{E}{(1-\nu^2)}[\nu\epsilon_x + \epsilon_y] \quad \text{--- (5)}$$

$$|||^{ly},\ \sigma_x = \frac{E}{(1-\nu^2)}[\nu\epsilon_y + \epsilon_x] \quad \text{--- (6)}$$

$$(3) \Rightarrow \tau_{xy} = \frac{E}{2(1+\nu)} \cdot \gamma_{xy} \times \frac{(1-\nu)}{(1-\nu)}$$

$\therefore \tau_{xy} = \frac{E(1-\nu)}{2(1-\nu^2)} \cdot \gamma_{xy}$ — (7)

Matrix eq^n of (5), (6) & (7)

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$\sigma = D.\,E$

Where, $D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\nu}{2}\right) \end{bmatrix}$

D- Strain transf^n matrix

$Q_3 = \frac{T_\infty \cdot k_3}{L_3} = \frac{100 \times 5}{200}$

$Q_3 = 2.5 / 2500$

$$\begin{bmatrix} Q_1 \\ 3.75 \\ 2.5 \\ Q_4 \end{bmatrix} = 10^3 \begin{bmatrix} & & \\ & 125 & -50 & \\ & -50 & 75 & \\ & & \end{bmatrix} \begin{bmatrix} 50 \\ T_2 \\ T_3 \\ 100 \end{bmatrix}$$

$3.75 = 10^3 (125 T_2 - 50 T_3)$

$2.5 = 10^3 (-50 T_2 + 75 T_3)$

$T_2 = 0.059 \times 10^3 = 59°C$

$T_3 = 0.072 \times 10^3 = 72°C$

Temp. displ. mat.

$$T = \begin{bmatrix} 50 \\ 59 \\ 72.7 \\ 100 \end{bmatrix}_{4\times1}$$

The heat flux at the enter and exist of the system

Step 5 :→

$$Q_1 = 10^3 [75 \times T_1 - 75 \times T_2]$$

$$= 10^3 (75 \times 50 - 75 \times 59)$$

$$Q_1 = -675 \times 10^3 \text{ watts}$$

$$Q_4 = 10^3 \times [-25 T_3 + 25 T_4]$$

$$= 10^3 \times [-25 \times 72.7 + 25 \times 100]$$

$$Q_4 = 682.5 \times 10^3 \text{ watts}$$

$$\begin{bmatrix} Q_1 \\ 0.24 \\ 0.0645 \\ Q_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & 6.2 & -1.4 & \\ & -1.4 & 5.7 & \\ & & & \end{bmatrix} \begin{bmatrix} 500 \\ T_2 \\ T_3 \\ 150 \end{bmatrix}$$

$$6.2 T_2 - 1.4 T_3 = 0.24$$

$$-1.4 T_2 + 5.7 T_3 = 0.0645$$

$$T_2 = 0.043$$

$$T_3 = 0.022$$

Temp displ. mat.

$$T = \begin{bmatrix} 500 \\ 0.043 \\ 0.0645 \\ 150 \end{bmatrix}$$

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1] Saint Venats Principle

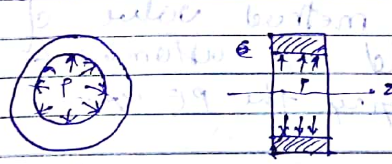
i) It states that as long as the different approximation are statically equivalent, the resulting soln will be valid provided we focus on region sufficiently far away from the support.

ii) We often have to make approximation in defining B.C to represent a support structure interface.

iii) for instance, consider a cantilever beam, free at one end & is ~~supported~~ attached to the column with rivet at the other end.

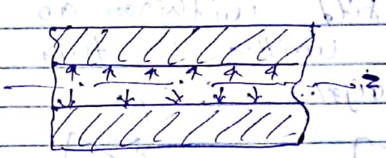
iv) Que arises as to whether the riveted joint is totally rigid or partially rigid.

Plane stress → A thin planar body subjected to in-plane loading on its edge surface is said to be in plane stress.



plane strain →

If a long body ~~is s.t a um~~ of a uniform C/S is s.to transverse loading along its length it is kn. a plane strain.



- Mathematical error.
- Discretization error.
- Roundoff error.

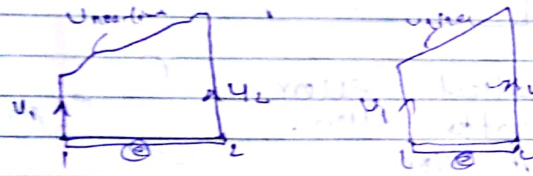
} error

- Rayleigh Ritz meth

i) Based on the fact that the total P.E could be used for finding an approximation soln for continuum.

- It is a mathematical tool used for finding the stress or displ. of pt. on body.
- In this method value of displ. the field is assumed to evaluate by applying the PE approach.

- Shape fn → It maps the value of DOF from nodes to pt. within the element.
- Once the S.F is defined the linear displ. field within an element can be written in terms of the nodal displ. q_1 & q_2.

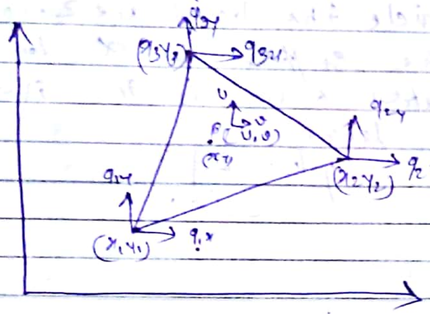


- Prop. of global stiff matrix
  1) The dim. of g.s.m is K is (N×N) where N is the no. of nodes. This follows from the fact that each node has 1 DOF.

- K is symmetric
- K is banded matrix i.e. that is all mat. outside the band are zero.
- The stiffness is ∝ to modulus of elasticity & area is inversely prop. to length

$k = \frac{AE}{L}$.

10) Strain displ. relation matrix (B) or K for CST using the fun.



The strain displ. is given by,

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{bmatrix} \quad \text{—(1).}$$

diff ~~w.r.t~~ u w.r.to natural co-ordinates (ξ, η) so by chain differentiation rule

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}.$$

matrix as

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \partial x/\partial \xi & \frac{\partial y}{\partial \xi} \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \end{bmatrix} \quad \text{—(2).}$$

$$J = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix}$$

$$= \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}_{2\times 2}$$

Jacob matrix is use to find out the area

$$A = \frac{1}{2} |J|$$

$$= \frac{1}{2} | x_{13} \cdot y_{23} - y_{13} \cdot x_{23} |$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

$$J^{-1} = \frac{Adj\,[J]}{|J|} = \frac{1}{|J|} Adj\,J.$$

$$= \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \cdot \begin{bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{bmatrix} q_{1x} - q_{3x} \\ q_{2x} - q_{3x} \end{bmatrix}$$

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23}(q_{1x}-q_{3x}) - y_{13}(q_{2x}-q_{3x}) \\ -x_{23}(q_{1x}-q_{3x}) + x_{13}(q_{2x}-q_{3x}) \end{bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{|J|}\left(q_{1x} y_{23} + q_{2x} y_{31} + q_{3x} y_{12}\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{|J|}\left[q_{1x} x_{32} + q_{2x} x_{13} + q_{3x} x_{21}\right]$$

Now diff v w.r.t. to not curr...

So

$$\frac{\partial v}{\partial x} = \frac{1}{|J|}\left(q_{1y} y_{23} + q_{2y} y_{31} + q_{3y} y_{12}\right]$$

$$\frac{\partial v}{\partial y} = \frac{1}{|J|}\left[q_{1y} x_{32} + q_{2y} x_{13} + q_{3y} x_{21}\right]$$

$$\epsilon = \frac{1}{|J|}\left[\quad\right]$$

$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times 6}$$

S.E at pt / vol. $= \frac{1}{2}\sigma^T \cdot \epsilon$

$= \frac{1}{2}\sigma^T \cdot \epsilon \cdot t \cdot dA$

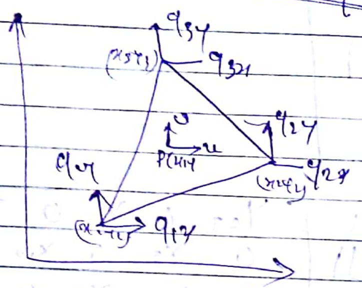
$= \frac{1}{2}\sigma^T \cdot \epsilon \cdot t \cdot A$

$= \frac{1}{2}(D \cdot \epsilon)^T \cdot \epsilon \cdot t \cdot A.$

$= \frac{1}{2}\epsilon^T D^T \epsilon \cdot t \cdot A$

$= \frac{1}{2} t \cdot A \cdot B^T \cdot D \cdot B.$

8) Shape fn for 2D CST / or Δ^{ry} ele



Let ξ, η be the natural coord.

Here the we can write disp at pt 'P' given

$u = N_1 q_{1x} + N_2 q_{2x} + N_3 q_{3x}$
$v = N_1 q_{1y} + N_2 q_{2y} + N_3 q_{3y}$

$N_1 = \xi$
$N_2 = \eta$
$N_3 = 1 - \xi - \eta$.

$u = \xi q_{1x} + \eta q_{2x} + (1-\xi-\eta) q_{3x}$
$v = \xi q_{1y} + \eta q_{2y} + (1-\xi-\eta) q_{3y}$

$u = \xi q_{1x} + \eta q_{2x} + q_{3x} - \xi q_{3x} - \eta q_{3x}$

$u = \xi(q_{1x} - q_{3x}) + \eta(q_{2x} - q_{3x}) + q_{3x}$

$v = \xi(q_{1y} - q_{3y}) + \eta(q_{2y} - q_{3y}) + q_{3y}$

Similarly we can write

$x = N_1 x_1 + N_2 x_2 + N_3 x_3$
$y = N_1 y_1 + N_2 y_2 + N_3 y_3$

~~$v = \xi (q_{1x} - q_{3x})$~~ $x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3$
$y = \xi(y_1 - y_3) + \eta(y_2 - y_3) + y_3$

Now specify natural coords for node (1, 2)

we know that

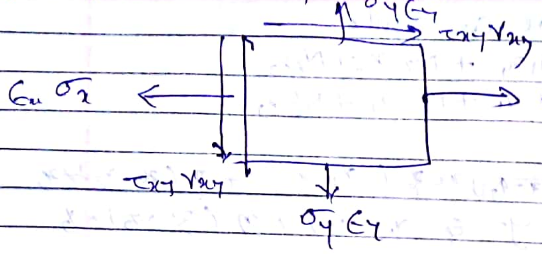
$N_1 + N_2 + N_3 = 1$

we know that

At node 1, $N_1 = 1$, $N_2\ N_3 = 0$
$N_2 = 1$ $N_1\ N_3 = 0$
$N_3 = 1$ $N_1\ N_2 = 0$

at node 1, $N_1 = \xi = 1$, $N_2 = \eta = 0$
$N_2 = \eta = 1$, $N_1 = \xi = 0$
$N_3 = \xi = 0$

a) Plain stress plain strain condn



Let σ_x, σ_y be the normal stress in x & y dirn

Let τ_{xy} be the shear stress in xy plane

of stress eqn ...

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Let $\epsilon_x\ \epsilon_y$ be the linear strain in x & y dir

be the strain that is given

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Now eq of strain dirn ... stress.

$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$G = \frac{E}{2(1+\mu)}$$

$$\gamma_{xy} = \frac{2(1+\mu)\cdot \tau_{xy}}{E}$$

Matrix eqn

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

~~Now ...~~ $\epsilon = C \cdot \sigma$

$$C = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix}$$

Now apply to plane ... plane stress

$$1 \times \mu = \mu \epsilon_x = \mu\left(\frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}\right)$$

$$\epsilon_y = \mu \frac{\sigma_y}{E} - \mu^2 \frac{\sigma_x}{E}$$

$$\mu \epsilon_x + \epsilon_y = \frac{\sigma_y}{E} - \mu^2 \frac{\sigma_y}{E}$$

$$\mu \epsilon_x + \epsilon_y = (1-\mu^2)\frac{\sigma_y}{E}$$

$$\sigma_y = \frac{E}{(1-\mu^2)}\left[\mu \epsilon_x + \epsilon_y\right]$$

$$\sigma_x = \frac{E}{1-\mu^2}\left[\mu \epsilon_y + \epsilon_x\right]$$

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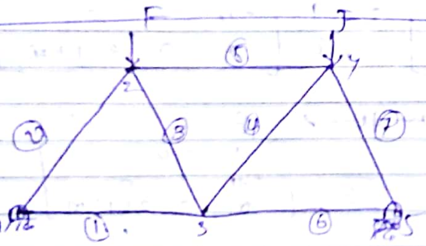
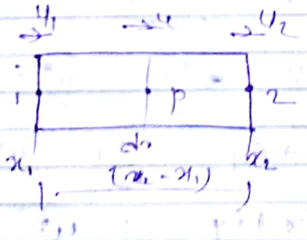
$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{E}{(1-\mu^2)}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \left(\frac{1-\mu}{2}\right)\end{bmatrix}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$$D = \frac{E}{(1-\mu^2)}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & \left(\frac{1-\mu}{2}\right)\end{bmatrix}$$

Truss



2- noded Shape fn



The strain at pt within an elm is given by

$$\epsilon = \frac{\partial u}{\partial x} \quad \text{—— (1)}$$

By chain differentiation rule we consider

$$\epsilon = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} \quad \text{—— (2)}$$

Where u = displ. at any pt

$$u = N_1 u_1 + N_2 u_2$$

where $N_1 = \frac{1-\xi}{2}$ $\quad N_2 = \frac{1+\xi}{2}$

$$= \left(\frac{1-\xi}{2}\right) u_1 + \left(\frac{1+\xi}{2}\right) u_2$$

diff w.r.to ξ

$$= \left(-\frac{u_1}{2} + \frac{u_2}{2}\right)$$

$$\frac{\partial u}{\partial \xi} = \frac{(u_2 - u_1)}{2} \quad \text{—— (3)}$$

We have $\xi = \frac{2}{(x_2 - x_1)}(x - x_1) - 1$

diff w.r.t x

$$\frac{\partial \xi}{\partial x} = \frac{2}{(x_2 - x_1)} \quad \text{—— (4)}$$

$$\epsilon = \frac{u_2 - u_1}{2} \times \frac{2}{x_2 - x_1}$$

$$\epsilon = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$\epsilon = B \cdot q$$

$$\left(B = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\right)$$